§ 8.1 cont'd.

Example. Find the length of the curve

$$y = \frac{x^2}{2} - \frac{\ln x}{4}, \qquad 2 \le x \le 4.$$

Soln. $L = \int_2^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \underbrace{\left(x - \frac{1}{4x}\right)^2}_{\left(\frac{dy}{dx}\right)^2}} = \sqrt{1 + \left(x^2 - \frac{1}{2} + \frac{1}{16x^2}\right)} = \sqrt{\left(x + \frac{1}{4x}\right)^2} = x + \frac{1}{4x}$$

since $2 \le x \le 4 \Rightarrow x > 0$ and so $x + \frac{1}{4x} > 0$.

$$\therefore L = \int_2^4 x + \frac{1}{4x}\, dx$$

$$= \left(\frac{x^2}{2} + \frac{1}{4}\ln x\right)\Big|_2^4 = \left(\frac{16}{2} + \frac{1}{4}\ln 4\right) - \left(\frac{4}{2} - \frac{1}{4}\ln 2\right)$$

do not need $|\cdot|$ since $x > 0$ if $2 \le x \le 4$.

$4 = 2^2 \Rightarrow \ln 4 = \ln 2^2 = 2\ln 2$

$$= 6 + \frac{2}{4}\ln 2 - \frac{1}{4}\ln 2$$

$$= 6 + \frac{1}{4}\ln 2$$

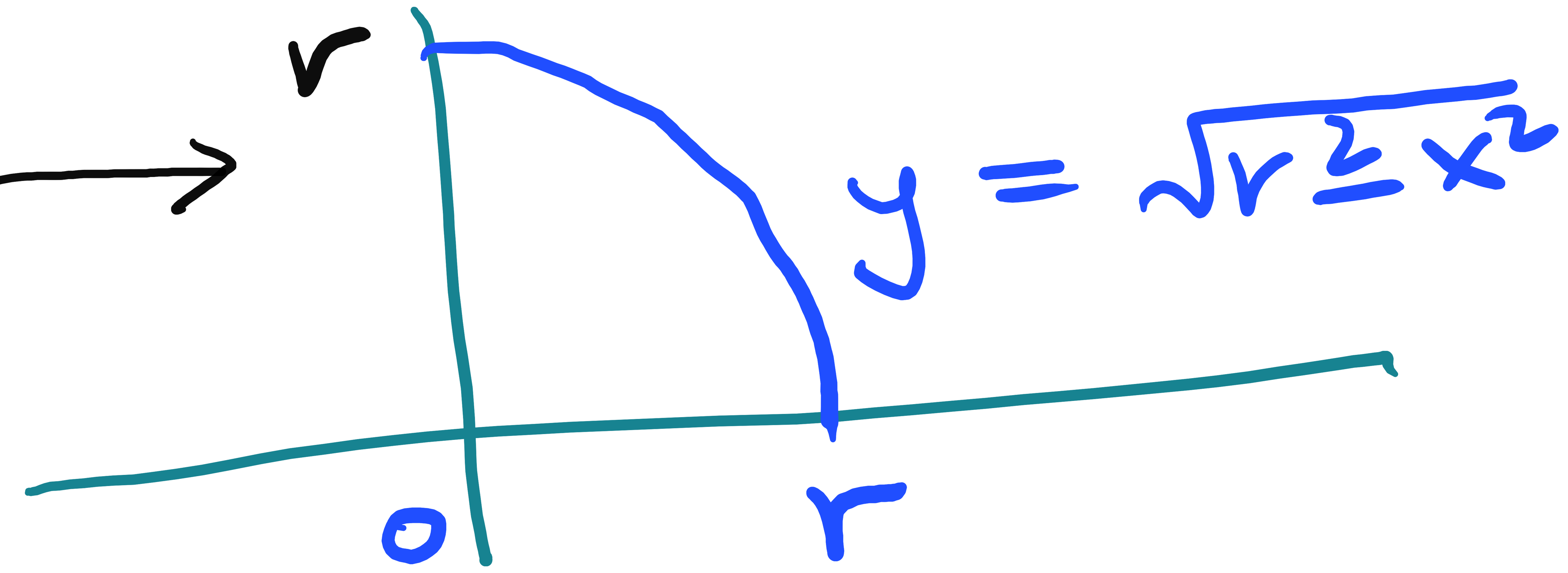
(no arbitrary constant since a definite integral)

Example Find the circumference of a circle.

$x^2 + y^2 = r^2 \Rightarrow y = \sqrt{r^2 - x^2}$ (equation of the top half of a circle).

We consider the portion in the first quadrant (then multiply by 4).

$$L = \int_0^r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$$\frac{dy}{dx} = \frac{1}{\cancel{2}}(r^2 - x^2)^{-1/2}(-\cancel{2}x) = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$L = \int_0^r \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\, dx$$

$$= \int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$

$$= \int_0^r \sqrt{\frac{r^2 - \cancel{x^2} + \cancel{x^2}}{r^2 - x^2}}\, dx$$ (common denominator)

$$= \int_0^r \frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$= \int_{x=0}^{r} \frac{r}{\sqrt{r^2-x^2}}\,dx$$

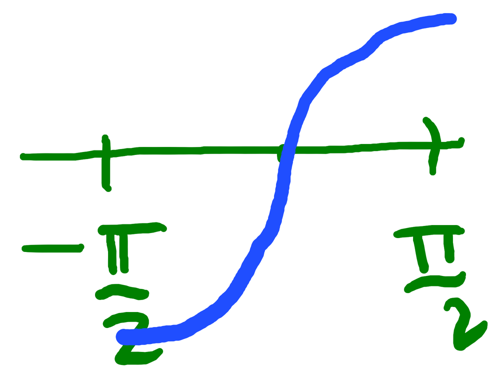
$\cos^2\theta = 1 - \sin^2\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

let $\boxed{x = r\sin\theta}$ $\quad$ $\boxed{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$

so that $\sin^{-1}\left(\frac{x}{r}\right)$ exists.



$\sin(\theta)$

$$r^2 - x^2 = r^2 - r^2\sin^2\theta$$
$$= r^2(1-\sin^2\theta)$$
$$= r^2\cos^2\theta$$

$$\boxed{dx = r\cos\theta\, d\theta}$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \frac{r \cdot r\cos\theta\, d\theta}{\sqrt{r^2\cos^2\theta}}$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \frac{r^2\cos\theta\, d\theta}{r\cos\theta} = \int_{\theta=0}^{\frac{\pi}{2}} r\, d\theta = r\theta\Big|_0^{\frac{\pi}{2}}$$

$$= r\frac{\pi}{2} - 0 = \frac{r\pi}{2}.$$

$\therefore$ The length of $\frac{1}{4}$ of arc of a circle is $\frac{r\pi}{2}$.
(i.e. the portion in the 1^{st} quadrant.)

$\therefore$ The circumference is
$$4\left(\frac{r\pi}{2}\right) = 2\pi r$$
as learned in high school.

§7.5. Strategies of Integration.

FIRST : Memorize "Table of Integration"

see Table on page 503 of Textbook (ed. 8)

i.e. §7.5.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

Table

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ed. 8.

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \quad a > 0$$

2 more at bottom of table are derived below.

$$I = \int \frac{1}{x^2-a^2}\,dx = \int \frac{1}{(x-a)(x+a)}\,dx$$

$$= \int \frac{A}{x-a} + \frac{B}{x+a}\,dx.$$

$$= \int \frac{\overset{A}{\overset{\shortparallel}{\left(\frac{1}{2a}\right)}}}{x-a} + \frac{\overset{B}{\overset{\shortparallel}{\left(-\frac{1}{2a}\right)}}}{x+a}\,dx$$

$$= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C$$

$$= \frac{1}{2a}\left(\ln\left|\frac{x-a}{x+a}\right|\right) + C$$

Determine A & B.

$A(x+a) + B(x-a) = 1.$

$x=a: \ 2aA = 1 \Rightarrow A = \frac{1}{2a}$

$x=-a: \ B(-2a) = 1$

$B = -\frac{1}{2a}$

OR equate coefficients.

coeff x: $A + B = 0 \Rightarrow A = -B$

1: $aA - aB = 1$

$\Rightarrow -aB - aB = 1$

$B = -\frac{1}{2a}$

$A = \frac{1}{2a}$

Next

$$I = \int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \ln\left| x + \sqrt{x^2 \pm a^2} \right| + C.$$

$$I = \int \frac{1}{\sqrt{x^2+a^2}}\,dx$$

$$\left(\begin{array}{l} \sin^2\theta + \cos^2\theta = 1 \\ \tan^2\theta + 1 = \sec^2\theta \end{array} \right)$$

Let $x = a\tan\theta$

$dx = a\sec^2\theta\,d\theta$.

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2} \quad // \sqrt{x^2+a^2}$$

$$= \sqrt{a^2(1+\tan^2\theta)}$$

$$= \sqrt{a^2\sec^2\theta}$$

$$= a\sec\theta.$$

$$I = \int \frac{1}{a\sec\theta}\, a\sec^2\theta\,d\theta$$

$$= \int \sec\theta\,d\theta$$

$$= \int \frac{1}{\cos\theta}\,\frac{\cos\theta}{\cos\theta}\,d\theta$$

$$= \int \frac{\cos\theta}{1-\sin^2\theta}\, d\theta \qquad u = \sin\theta, \quad du = \cos\theta\, d\theta$$

$$= \int \frac{1}{1-u^2}\, du = -\int \frac{1}{u^2-1}\, du$$

(We just did this in the previous integral. $\int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$, $x = u$, $a = 1$)

$$= -\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|^{-1} + C$$

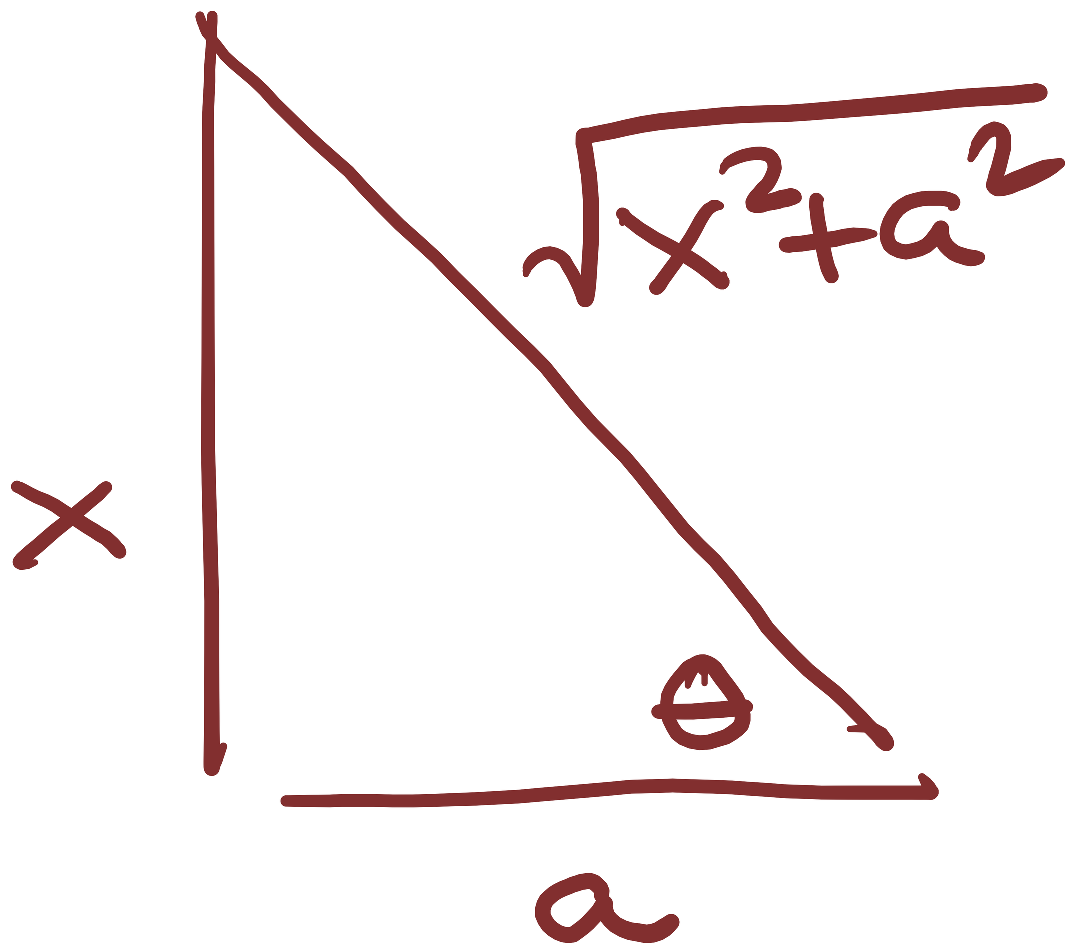
$$= \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + C.$$

Back to original variables.

$u = \sin\theta$.

$$= \frac{1}{2}\ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right| + C$$

and $x = a\tan\theta$.



$$x = a\tan\theta$$

$$\tan\theta = \frac{x}{a}$$

$$\sin\theta = \frac{x}{\sqrt{x^2+a^2}}.$$

$$I = \frac{1}{2}\ln\left|\frac{\dfrac{x}{\sqrt{x^2+a^2}}+1}{\dfrac{x}{\sqrt{x^2+a^2}}-1}\right| + C$$

(Common denominator of $\sqrt{x^2+a^2}$ in numerator & denominator)

$$= \frac{1}{2}\ln\left|\frac{x+\sqrt{x^2+a^2}}{x-\sqrt{x^2+a^2}}\right| + C.$$

$$= \ln\left|\frac{(x+\sqrt{x^2+a^2})(x+\sqrt{x^2+a^2})}{(x-\sqrt{x^2+a^2})(x+\sqrt{x^2+a^2})}\right|^{\frac{1}{2}} + C$$

$$= \ln \left| \frac{\left(x+\sqrt{x^2+a^2}\right)^2}{\cancel{x^2} - (\cancel{x^2}+a^2)} \right|^{1/2} + C$$

$$= \ln \left| \frac{x+\sqrt{x^2+a^2}}{a} \right| + C$$

$$= \ln \left| x+\sqrt{x^2+a^2} \right| \underbrace{- \ln a + C}_{K}.$$

$$= \ln \left| x+\sqrt{x^2+a^2} \right| + K.$$

if C is an arbitrary constant then $C - \ln a$ is an arbitrary constant.