§7.4 Partial Fraction Decomposition, cont'd.

Example (Distinct linear factors, cont'd from last lecture).

$$I = \int \frac{3x+2}{x(2x+1)}\,dx = \int \frac{3x+2}{x(2x+1)}\,dx$$

$$= \int \frac{A}{x} + \frac{B}{2x+1}\,dx,$$ A, B to be determined.

$$\frac{3x+2}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}.$$

Put RHS over a common denominator (i.e. the denominator on the LHS) and equate numerators.

$$= \frac{A(2x+) + Bx}{x(2x+1)}$$

$$\therefore \boxed{A(2x+1) + Bx = 3x+2} \Rightarrow (2A+B)x + A = 3x+2$$

coeff. x : $2A + B = 3$

coeff: 1 : $A = 2$

$A = 2 \Rightarrow 4 + B \quad 3 \Rightarrow B = -1$.

$$I = \int \frac{2}{x} + \frac{-1}{2x+1}\,dx$$

$$= 2\ln|x| - \tfrac{1}{2}\ln|2x+1| + C.$$

$$\text{OR} = \ln\left(\frac{x^2}{\sqrt{|2x+1|}}\right) + C$$

Example $I = \int \frac{1}{x^2-a^2}\,dx, \quad a \neq 0.$

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}.$$

$$A(x+a) + B(x-a) = 1 \quad (*)$$

Method 1. let $x = -a$ in $(*)$ $\Rightarrow$ $B(-2a) = 1$

$$B = -\frac{1}{2a}$$

$x = a$ in $(*)$ $\Rightarrow$ $A(2a) = 1$

$$A = \frac{1}{2a}$$

OR Method 2. Equate coefficients

coeff x : $A + B = 0 \implies A = -B$

coeff 1 : $Aa - Ba = 1 \implies -Ba - Ba = 1$

$B = -\frac{1}{2a}. \implies A = \frac{1}{2a}$

$$I = \int \frac{1}{x^2-a^2}\,dx = \int \frac{A}{x-a} + \frac{B}{x+a}\,dx$$

$$= \int \frac{(1/2a)}{x-a} + \frac{(-1/2a)}{x+a}\,dx$$

$$= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C.$$

Case 2 $Q(x)$ is a product of linear factors, some of them repeated.

i.e. $(ax+b)^n$ $n>1$ occurs in $Q(x)$

Include $\underline{n}$ terms: $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$.

Example: $I = \int \frac{1}{(x-1)^2(x+4)}\,dx = \int \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+4)}\,dx$

$$\frac{1}{(x-1)^2(x+4)} = \frac{A(x-1)(x+4) + B(x+4) + C(x-1)^2}{(x-1)^2(x+4)}$$

Common denominator (SAME on both sides) . .
Equate numerators.

$$A(x-1)(x+4) + B(x+4) + C(x-1)^2 = 1. \quad \textcircled{*}$$

$$A(x^2+3x-4) + B(x+4) + C(x^2-2x+1) = 1.$$

coeff x^2 : $A \quad\quad + C = 0$ ①

Coeff x : $3A \quad +B \quad -2C = 0$ ②

Coeff 1 : $-4A \quad +4B \quad +C = 1$ ③

SOLVE this system OR

Substitute values of the roots of the linear terms in (*)

Let $x=1$ in (*) $\quad 0 + 5B + 0 = 1 \Rightarrow B = \frac{1}{5}$.

Let $x=-4$ in (*) $\quad 0 + 0 + 25C = 1 \Rightarrow C = \frac{1}{25}$.

Put $C = \frac{1}{25}$ in (1) $\rightarrow A = -C = -\frac{1}{25}$.

$$I = \int \frac{-\frac{1}{25}}{x-1} + \frac{\frac{1}{5}}{(x-1)^2} + \frac{\frac{1}{25}}{x+4}\, dx$$

$$= -\frac{1}{25}\ln|x-1| - \frac{1}{5}(x-1)^{-1} + \frac{1}{25}\ln|x+4| + C.$$

<u>Case</u> 3 $\quad Q(x)$ has irreducible quadratic factors (<u>not</u> repeated).

If $Q(x)$ has a factor: ax^2+bx+c, when $b^2-4ac<0$.

Include a term: $\dfrac{Ax+B}{ax^2+bx+c}$.

To integrate $\int \frac{Ax+B}{ax^2+bx+c}\,dx$, complete the square and use $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$

Example

$$\frac{27x^2}{(x-1)(x^2+1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+x+1}.$$

(x^2+1 & x^2+x+1 are both irreducible quadratic factor)

Example $I = \int \frac{x^2+5}{x^2-x+2}\,dx.$

$\deg(P(x)) = \deg(Q(x))$ $\therefore$ divide

$$\begin{array}{r} 1 \\ x^2-x+2 \,\big)\, x^2 \qquad\quad +5 \\ x^2 - x \quad +2 \\ \hline x \quad +3 \end{array}$$

$x+3 = R(x)$ (remainder)

$$I = \int 1 + \frac{x+3}{x^2-x+2}\,dx$$

ax^2+bx+c

$a=1 \quad b=-1 \quad c=2.$

(x^2-x+2 is an irreducible quadratic)

b^2-4ac

$= 1-4(1)(2) = -7 < 0$

I is already in its partial fraction decomposition: n irreducible quadratic.

$A = 1, \quad B = 3$

$$\frac{Ax+B}{x^2-x+2} = \frac{x+3}{x^2-x+2}$$ (Nothing to do.)

$$\int 1 + \frac{x+3}{x^2-x+2}\,dx$$

Complete the square.

$(x-\frac{1}{2})^2 + \frac{7}{4}$.

$$= \int 1 + \frac{x+3}{(x-\frac{1}{2})^2+\frac{7}{4}}\,dx$$

Let $\begin{cases} u = x - \frac{1}{2} \Rightarrow x = u + \frac{1}{2} \\ du = dx \end{cases}$

$$= x + \int \frac{u+\frac{1}{2}+3}{u^2+7/4}\,du$$

$$= x + \int \frac{u}{u^2+\frac{7}{4}}\,du + \frac{7}{2}\int \frac{1}{u^2+\frac{7}{4}}\,du$$

$a = \sqrt{\frac{7}{4}}$.

$\left(\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.\right)$

$$= x + \frac{1}{2}\ln\left(u^2+\frac{7}{4}\right) + \frac{7}{2}\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{u}{\sqrt{7}/2}\right) + C$$

$$= x + \frac{1}{2}\ln\left(\left(x-\frac{1}{2}\right)^2+\frac{7}{4}\right) + \sqrt{7}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{7}/2}\right) + C$$

$$x + \frac{1}{2}\ln(x^2-x+2) + \sqrt{7}\tan^{-1}\left(\frac{2x-1}{\sqrt{7}}\right) + C$$