§7.3 Trig Subst. cont'd.

Example cont'd from last lecture:

$$I = \int \frac{1}{\sqrt{4x^2-4x-7}}\,dx$$

$$= \int \frac{1}{\sqrt{(2x-1)^2-8}}\,dx \quad \text{(by Completing the Square)}$$

Trig subst. $2x-1 = \sqrt{8}\sec\theta$, $\begin{cases} 0 \le \theta < \frac{\pi}{2} \\ \text{OR} \\ \pi \le \theta < \frac{3\pi}{2} \end{cases}$

$\therefore\ 2\,dx = \sqrt{8}\sec\theta\tan\theta\,d\theta.$

$$I = \int \frac{1}{\sqrt{(\sqrt{8}\sec\theta)^2-(\sqrt{8})^2}}\,\frac{\sqrt{8}}{2}\sec\theta\tan\theta\,d\theta$$

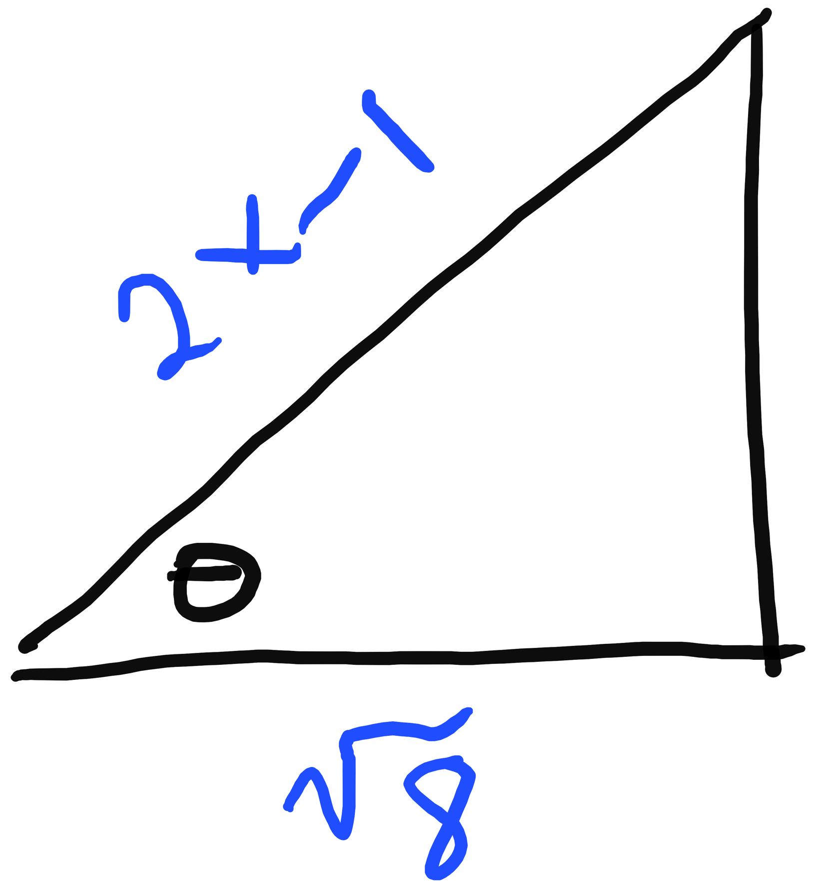
$$= \frac{1}{2}\int \frac{1}{\cancel{\sqrt{8}}\sqrt{\sec^2\theta-1}}\,\frac{\cancel{\sqrt{8}}}{2}\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{2}\int \frac{1}{\cancel{\sqrt{\tan^2(\theta)}}}\sec\theta\,\cancel{\tan\theta}\,d\theta$$

$$= \frac{1}{2}\int \sec\theta \, d\theta$$

$$\therefore \ I = \frac{1}{2} \ln\left|\sec\theta + \tan\theta\right| + C$$

RETURN to ORIGINAL VARIABLES.



Our substitution was
$2x-1 = \sqrt{8}\sec\theta$

$$\Rightarrow \sec\theta = \frac{2x-1}{\sqrt{8}}$$

$$\Rightarrow \cos\theta = \frac{\sqrt{8}}{2x-1}$$

Pythagoras

$$\sqrt{(2x-1)^2-8}$$

$$= \frac{1}{2} \ln\left| \frac{2x-1}{\sqrt{8}} + \frac{\sqrt{(2x-1)^2-8}}{\sqrt{8}} \right| + C.$$

OR

$$= \frac{1}{2} \ln\left| 2x-1 + \sqrt{(2x-1)^2-8} \right| - \frac{1}{2}\ln\sqrt{8} + C.$$

OR $\frac{1}{2} \ln\left|2x-1 + \sqrt{(2x-1)^2-8}\right| + K$ arb. constant

Since C is an arbitrary constant, any constant can be incorporated in C. It will still be an arbitrary constant.

Example. Definite Integral.

$$I = \int_0^1 \sqrt{x^2+1}\, dx$$

Let $x = \tan(\theta)$ $\qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$dx = \sec^2\theta\, d\theta$$

Change the limits of integration.

$$x = 0 \Rightarrow \theta = 0$$

$$x = 1 \Rightarrow \theta = \frac{\pi}{4}.$$

$$I = \int_{\theta=0}^{\frac{\pi}{4}} \sqrt{\tan^2\theta + 1}\, \sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2\theta}\, \sec^2\theta\, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$$

Usually, when odd power of sec or even power of tan, express the integrand entirely in terms of sec and ~~use~~ $\int$ by parts.

let $\quad u = \sec\theta \qquad\qquad v = \tan\theta$

$\qquad du = \sec\theta\tan\theta\, d\theta \qquad dv = \sec^2\theta\, d\theta$

$$I = \sec\theta\tan\theta\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^2\theta\sec\theta\, d\theta.$$

$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \underbrace{(\sec^2\theta - 1)\sec\theta}_{\to\ \sec^3\theta}\, d\theta$$

$$2I = 2\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta = \sec\theta\tan\theta\Big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \sec\theta\, d\theta$$

$$\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta = \frac{1}{2}\left(\sec\theta\tan\theta\Big|_0^{\frac{\pi}{4}} + \ln|\sec\theta + \tan\theta|\,\Big|_0^{\frac{\pi}{4}}\right)$$

$$= \frac{1}{2}\Big((\sqrt{2})(1) - (1)(0) + \big[\ln|(\sqrt{2}+1)| - \ln(1+0)\big]\Big)$$

$$= \frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)$$

§7.4 Partial Fraction Decomposition.

– for integration of rational functions

i.e $\int \frac{P(x)}{Q(x)}\,dx$ where $P(x)$ & $Q(x)$ are polynomials.

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$,

$a_n \neq 0$.

$\underbrace{\deg}_{\text{degree}}(P(x)) = n$

<u>IDEA</u>. Express $P(x)/Q(x)$ as a sum of simpler functions that we can integrate.

i.e.

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C,$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

$$\int \frac{x}{x^2 \pm a}\,dx = \frac{1}{2}\ln|x^2 \pm a| + C.$$ (use subst $u = x^2 \pm a$.

n pos integer. $$\int \frac{1}{(ax \pm b)^n}\,dx = \left(\frac{1}{a}\right)\left(\frac{1}{-n+1}(ax \pm b)^{-n+1}\right) + C.$$

(use subst. $u = ax \pm b$.)

$n \geq 2$. pos integer. $$\int \frac{1}{(x^2+a^2)^n}\,dx$$ use trig subst $x = a\tan\theta$.

$n \geq 2$ pos integer $$\int \frac{x}{(x^2+a^2)^n}\,dx$$ use subst. $u = x^2 + a^2$.

STEP 1 If $\deg(P) \geq \deg(Q)$, then divide Q into P until a remainder R with $\deg(R(x)) < \deg(Q(x))$.

Thm $\dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$, S, R, Q are polynomials

Example $\displaystyle\int \frac{x^2+1}{x+1}\,dx$

$P(x) = x^2+1$

$Q(x) = x+1$

$\deg(P(x)) > \deg(Q(x))$

$$\begin{array}{r|l} & S(x) \\ \hline Q(x) & P(x) \\ & \vdots \\ \hline & R(x) \end{array}$$

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

$$\begin{array}{r|l} & x \quad - 1 \\ \hline x+1 & x^2 \qquad + 1 \\ & x^2 + x \\ \hline & \quad -x + 1 \\ & \quad -x - 1 \\ \hline & \qquad 2 \leftarrow R(x) \end{array}$$

$$\frac{P(x)}{Q(x)} = \frac{x^2+1}{x+1} = x-1 + \frac{2}{x+1}$$

$$\int \frac{x^2+1}{x+1}\,dx = \int x - 1 + \frac{2}{x+1}\,dx$$

$$= \frac{x^2}{2} - x + 2\ln|x+1| + C.$$

STEP 2 Factor $Q(x)$ into a product of linear factors $(ax+b)$ and IRREDUCIBLE quadratic factors (ax^2+bx+c where $b^2-4ac<0$)

eg. $Q(x) = x^4 - 1 = (x^2+1)(x^2-1)$

$$= \underbrace{(x^2+1)}_{\text{IRREDUCIBLE quadratic}}(x-1)(x+1).$$

$b^2 - 4ac = -4.$

$0 \quad 1 \quad 1$

Quadratic formula

$ax^2+bx+c=0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

If $b^2-4ac=0$ then ax^2+bx+c is an IRREDUCIBLE QUADRATIC

STEP 3: Express $\frac{R(x)}{Q(x)}$, where $\deg(R(x)) < \deg(Q(x))$.

as a sum of "partial fractions".

Case 1. $Q(x)$ is a product of distinct linear factors.

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots\cdots(a_nx+b_n)$$

Then,

$$\frac{R(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$

where $A_1, A_2, \ldots, A_n$ are to be determined.

Example $I = \int \frac{3x+2}{2x^2+x}\,dx$

$\deg(P(x)) = 1$
$\deg(Q(x)) = 2$
$P(x) = R(x)$

$$= \int \frac{3x+2}{x(2x+1)}\,dx$$ ← distinct linear factors

$x = 1x + 0$, $a_1 = 1$, $b_1 = 0$

$2x+1 = a_2 x + b_2$, $a_2 = 2$, $b_2 = 1$

$$\frac{3x+2}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}$$

A, B to be determined.

Put the RHS over a common denominator and equate numerators.

$$\frac{3x+2}{x(2x+1)} = \frac{A(2x+1) + Bx}{x(2x+1)}$$

$$\therefore\ 3x+2 = A(2x+1) + Bx.$$ (find A & B)

TO BE CONTINUED NEXT LECTURE.