re: § 7.2.

$$\int \tan^2(x)\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C.$$

2 Methods: $\int \tan^{2k+1}(x)\,dx$

Method 1:

$$= \int \tan^{2k-1}(x)\tan^2(x)\,dx$$

$$= \int \tan^{2k-1}(x)(\sec^2 x - 1)\,dx$$

$$= \int \tan^{2k-1}(x)\sec^2 x\,dx - \int \tan^{2k-1}(x)\,dx$$

(Reduction formula).

let $u = \tan x$

$du = \sec^2 x\,dx$

$$= \int u^{2k-1}\,du - \int \tan^{2k-1}(x)\,dx$$

$$= \frac{u^{2k}}{2k} - \int \tan^{2k-1}(x)\,dx$$

$$\int \tan^{2k+1}(x)\,dx = \frac{\tan^{2k}(x)}{2k} - \int \tan^{2k-1}(x)\,dx$$

$\vdots$

$$\int \tan^3(x)\,dx = \int \tan x(\sec^2-1)\,dx$$

$$= -\int \tan x\,dx + \int \tan x \sec^2 x\,dx$$

$''$

$$= -\ln|\sec x| + \frac{\tan^2 x}{2} + C$$

Method 2.

$$\int \tan^9(x)\,dx = \int \frac{\tan^9 x}{\sec^2 x} \sec^2 x\,dx$$

let $u = \tan x$

$du = \sec^2 x\,dx$

$\sec^2 x = (\tan^2 x + 1)$

$$= \int \frac{u^9}{u^2+1}\,du$$

Polynomial Division:

$$
\begin{array}{r|l}
 & u^7 - u^5 + u^3 - u \\
\hline
u^2+1 & u^9 \\
 & \underline{u^9 + u^7} \\
 & -u^7 \\
 & \underline{-u^7 - u^5} \\
 & u^5 \\
 & \underline{u^5 + u^3} \\
 & -u^3 \\
 & \underline{-u^3 - u} \\
 & \underline{u} \leftarrow \text{Remainder}
\end{array}
$$

$$\therefore \frac{u^9}{u^2+1} = u^7 - u^5 + u^3 - u + \frac{u}{u^2+1}$$

$$\int \frac{u^9}{u^2+1}\,du = \int (u^7 - u^5 + u^3 - u)\,du + \int \frac{u}{u^2+1}\,du$$

$$w = u^2+1$$
$$dw = 2u\,du$$

$$= \frac{u^8}{8} - \frac{u^6}{6} + \frac{u^4}{4} - \frac{u^2}{2} + \frac{1}{2}\int \frac{1}{w}\,dw$$

$$= \frac{u^8}{8} - \frac{u^6}{6} + \frac{u^4}{4} - \frac{u^2}{2} + \frac{1}{2}\ln|w| + C.$$

$$= \frac{\tan^8(x)}{8} - \frac{\tan^6(x)}{6} + \frac{\tan^4(x)}{4} - \frac{\tan^2(x)}{2} + \frac{1}{2}\ln(u^2+1) + C$$

$$= \frac{\tan^8(x)}{8} - \frac{\tan^6(x)}{6} + \frac{\tan^4(x)}{4} - \frac{\tan^2(x)}{2} + \frac{1}{2}\ln(\tan^2(x)+1) + C$$

§ 7.3. Trig Substitution.

Inverse substitution. (reverse subst).

$$\int f(x)\,dx = \int f(g(t))\,dg(t) = \int f(g(t))\,g'(t)\,dt.$$

let $x = g(t)$

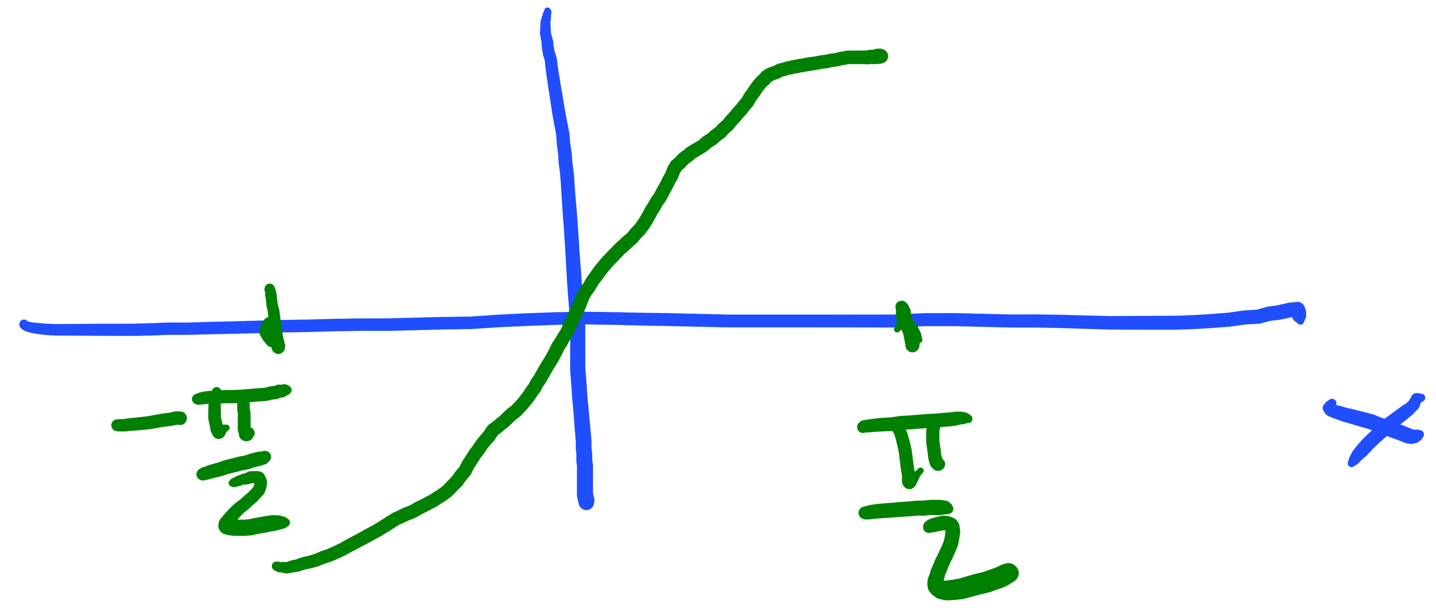
$g(t)$ must have an inverse

$$t = g^{-1}(x).$$

Example. Evaluate $I = \int \sqrt{16 - x^2}\,dx.$

Let $x = 4\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = 4\cos\theta\, d\theta$



$$I = \int \sqrt{16-x^2}\,dx = \int \sqrt{16-16\sin^2\theta}\ \underbrace{4\cos\theta\, d\theta}_{dx.}$$

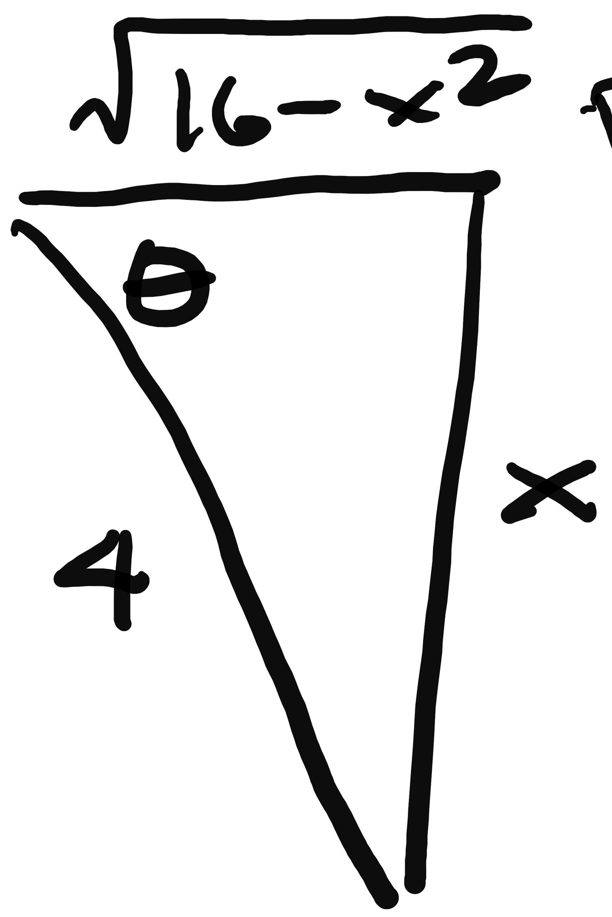
$$= 4\int \sqrt{1-\sin^2\theta}\ 4\cos\theta\, d\theta$$

$$= 16\int \cos^2\theta\ d\theta.$$

$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

$$= \frac{16}{2}\int 1+\cos(2\theta)\, d\theta$$

$$= 8\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$



Pythagorus.

$x = 4\sin\theta$

$\sin\theta = x/4 = \frac{\text{opp.}}{\text{hyp.}}$

RETURN TO ORIGINAL VARIABLES.

$\Rightarrow \theta = \underline{\sin^{-1}\left(\frac{x}{4}\right)}$

$$= 8\left(\sin^{-1}\left(\frac{x}{4}\right) + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right) + C$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= 8\left(\sin^{-1}\left(\frac{x}{4}\right) + \underbrace{\frac{x}{4}}_{\sin\theta}\underbrace{\left(\frac{\sqrt{16-x^2}}{4}\right)}_{\cos\theta}\right) + C.$$

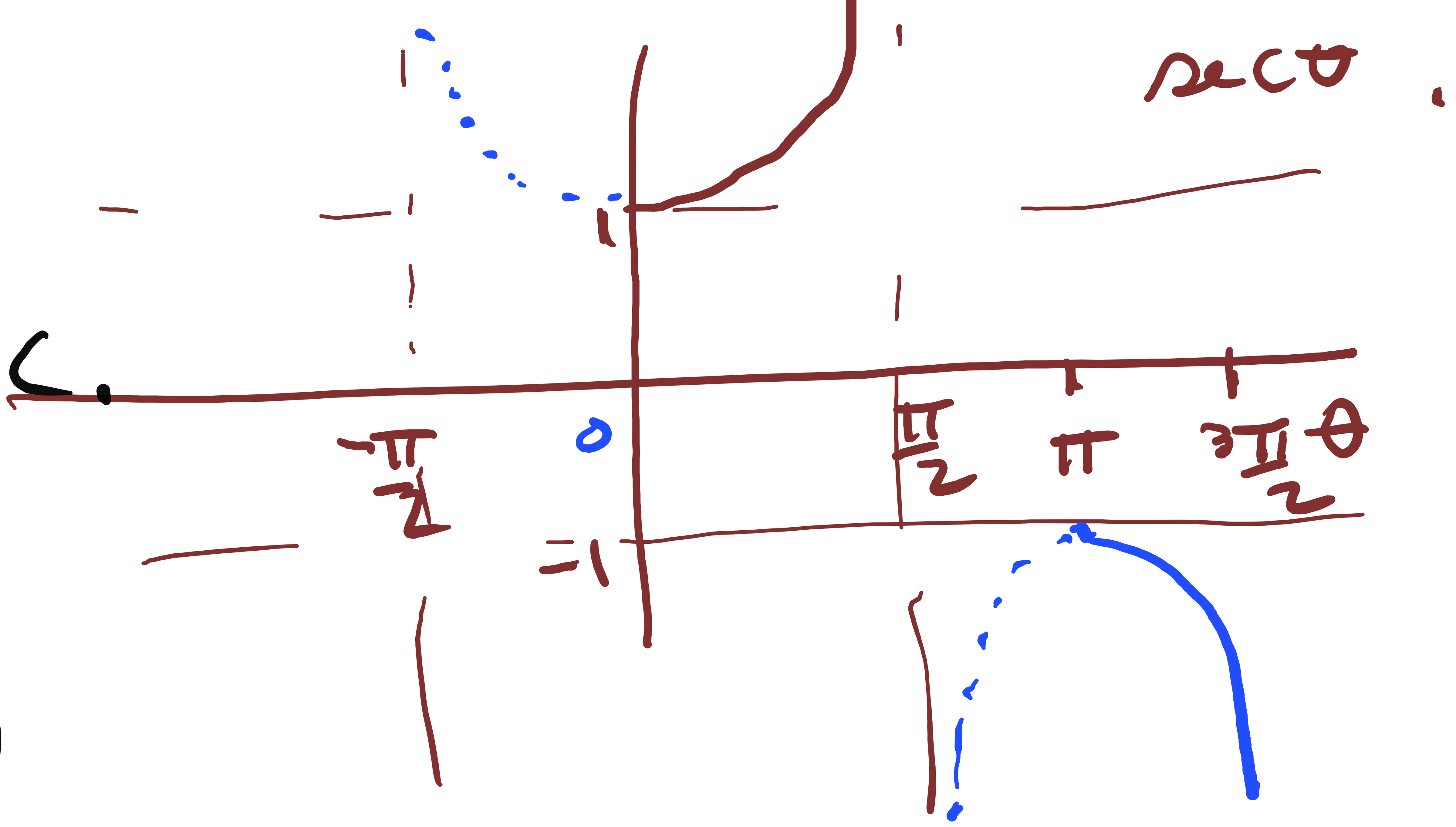
	Subst.	Interval	Use Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta.$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$0 \le \theta < \frac{\pi}{2}$ OR $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

$\sqrt{a^2+x^2} = \sqrt{a^2(1+\tan^2(x))} = \sqrt{a^2\sec^2\theta}$

Memorize.

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C.$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right).$$



Completing the Square.

$$ax^2+bx+c = \underbrace{\left(\sqrt{a}\,x+\frac{b}{2\sqrt{a}}\right)^2}_{ax^2+bx+\frac{b^2}{4a}} - \frac{b^2}{4a}+c$$

Example: $(4x^2-4x-7)$

$$= (2x-1)^2-8$$

Example $I = \int \frac{1}{\sqrt{4x^2-4x-7}}\,dx.$

$$= \int \frac{1}{\sqrt{(2x-1)^2-8}}\,dx$$ (Completing the square).

"Reverse" Trig subst.

let $2x-1 = \sqrt{8}\sec\theta$ $\quad 0 \le \theta < \frac{\pi}{2}$ OR $\pi \le \theta < \frac{3\pi}{2}$.

$\underline{2\,dx} = \sqrt{8}\sec\theta\tan\theta\,d\theta$

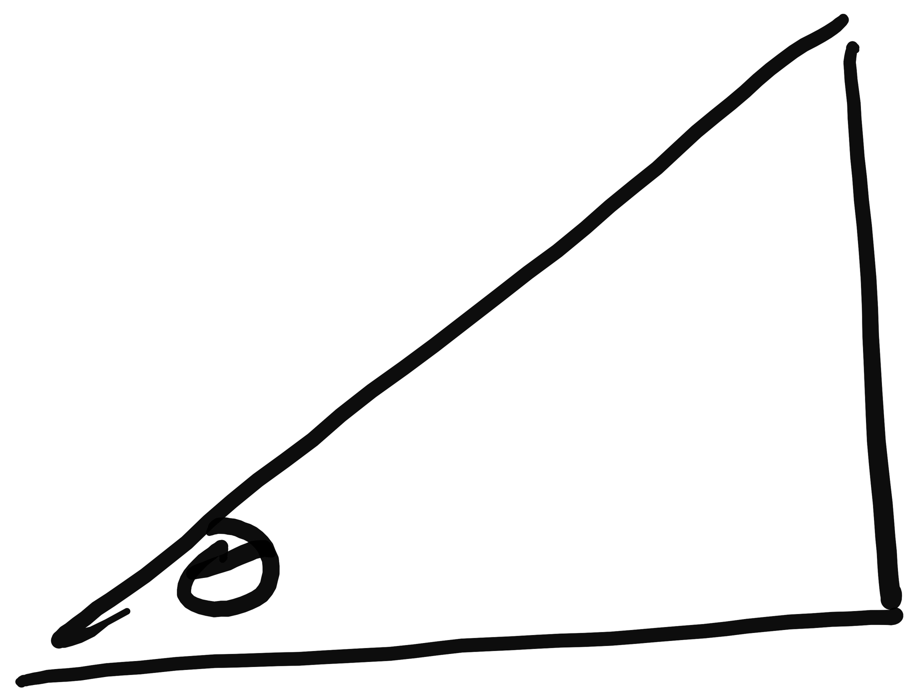
$$I = \int \frac{1}{\sqrt{8\sec^2\theta-8}}\underbrace{\left(\frac{\sqrt{8}}{2}\sec\theta\tan\theta\,d\theta\right)}_{dx}.$$

$$= \int \frac{1}{\cancel{\sqrt{8}}}\,\frac{1}{\cancel{\sqrt{\tan^2\theta}}}\,\frac{\cancel{\sqrt{8}}}{2}\sec\theta\,\cancel{\tan\theta}\,d\theta.$$

$$= \frac{1}{2} \int \sec\theta \, d\theta$$

$$= \frac{1}{2} \ln|\sec\theta + \tan\theta| + C.$$

RETURN TO ORIG VAR.



$$2x-1 = \sqrt{8}\sec\theta$$

TO BE CONT'D.