§ 7.2 Trig Integrals

(Need to know your trig identities).

(a) Power of $\cos(x)$ is odd in

$$I = \int \sin^m(x) \cos^{2k+1}(x)\,dx \qquad (m, k, \text{ non neg. integers})$$

$$= \int \sin^m(x) \left(\cos^2(x)\right)^k \underline{\cos(x)\,dx}$$

- save one $\cos(x)$ for dx.
- subst $\underline{u = \sin(x)}$ so $\underline{du = \cos(x)\,dx}$
- use $\cos^2(x) = 1 - \sin^2(x)$.

$$= \int \sin^m(x) \left(1 - \sin^2(x)\right)^k \cos(x)\,dx$$

$$= \int u^m (1 - u^2)^k \,du$$

Expand using the BINOMIAL Thm.

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}h^3 + \cdots + nxh^{n-1} + h^n$$

$$= x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \dots + \binom{n}{n-1}xh^{n-1} + \binom{n}{n}h^n.$$

where $\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$

Example: $(x+h)^3 = x^3 + \binom{3}{1}x^2h + \binom{3}{2}xh^2 + \binom{3}{3}h^3 = x^3 + 3x^2h + 3xh^2 + h^3.$

Example. $\int \sin^2 x \cos^3 x \, dx$, $\quad m=2, \; k=1, \; 2k+1=3.$

$$= \int \sin^2 x \, (\cos^2 x)^1 \cos x \, dx$$

$$= \int \sin^2 x \, (1 - \sin^2 x) \cos x \, dx$$

$$= \int (\sin^2 x - \sin^4 x) \cos x \, dx$$

$$\begin{cases} u = \sin x \\ du = \cos x \, dx \end{cases}$$

$$= \int (u^2 - u^4) \, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

(b) Power of $\sin(x)$ is odd.

$$I = \int \sin^{2k+1}(x)\cos^{n}(x)\,dx$$

Similarly save one $\sin(x)$ for dx

subst $u = \cos(x)$

$du = -\sin(x)\,dx$

use $\sin^2(x) = 1 - \cos^2 x$.

If both powers are odd use (a) or (b).

(c) If powers of both sin and cos are EVEN.

use $\frac{1}{2}$ angle formulas

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \quad (1)$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x)) \quad (2)$$

and. $$\sin(x)\cos(x) = \frac{1}{2}\sin(2x). \quad (3)$$

Example: $\int_0^{\pi} \cos^2 x \sin^2 x \, dx$

using (3)

$$= \int_0^{\pi} (\sin x \cos x)^2 \, dx$$

$$= \int_0^{\pi} (\sin x \cos x)^2 \, dx$$

$$= \int_0^{\pi} \left(\tfrac{1}{2} \sin(2x)\right)^2 dx .$$

$$= \frac{1}{4} \int_0^{\pi} \sin^2(2x) \, dx$$

use (1) .

$$= \frac{1}{4} \int_0^{\pi} \frac{1}{2} \left(1 - \cos(4x)\right) dx$$

by (1)

$$= \frac{1}{8} \left(x - \frac{\sin(4x)}{4} \right\rvert_0^{\pi}$$

$$= \frac{1}{8} \left((\pi - 0) - (0 - 0) \right) = \frac{\pi}{8} .$$

$$\int \tan^m x \sec^n x \, dx.$$

(a) If the power of sec. is EVEN $(n=2k, k \geq 1)$

- save a factor of $\sec^2 x$ for the dx.
- use $\sec^2 = 1 + \tan^2 x$.
- subst. $u = \tan x$

$$I = \int \tan^m x \sec^{2k} x \, dx$$

$$= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx$$

Let $u = \tan x$

$du = \sec^2 x \, dx$ use $\sec^2 = 1 + \tan^2 x$

$$= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int u^m (1 + u^2)^{k-1} \, du$$

Expand by multiplying or using the Binomial Thm. Then go back to original variable.

Example. $I = \int \tan^2 x \sec^2 x \, dx.$

$u = \tan x$ $\qquad = \int u^2 \, du$

$du = \sec^2 x \, dx$

$$= \frac{u^3}{3} + C = \frac{\tan^3 x}{3} + C.$$

(b) If the power of tan is <u>ODD</u>. $(m = 2k+1)$.

- save a factor of $\sec x \tan x$
- use $\tan^2 x = \sec^2 x - 1$
- subst $u = \sec x$.

$$I = \int \tan^{2k+1} x \sec^n x \, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$I = \int (\tan^2 x)^k (\sec^{n-1} x) \underline{(\sec x \tan x)} \, dx$$

$$= \int (\sec^2 x - 1)^k (\sec^{n-1}(x)) (\sec x \tan x)\, dx$$

$$= \int (u^2 - 1)^k (u^{n-1})\, du.$$

- expand & integrate
go back to the original variables.

Other cases: not so clear.

NOTE: $\int \tan x \, dx = \ln |\sec x| + C.$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C.$$

EVEN power of tan
OR
ODD power of sec

Try (doesn't always help)
expressing the integrand entirely in terms of sec x
+ Try integrating by parts.

Note: $\int \cot^m x \csc^n x \, dx$.

use $1 + \cot^2 x = \csc^2 x$

and proceed as for $\underline{\tan^m(x) \sec^n(x)}$

Consider

(a) $\int \sin(mx) \cos(nx) \, dx$

(b) $\int \sin(mx) \sin(nx) \, dx$

(c) $\int \cos(mx) \cos(nx) \, dx$.

Use the Trig Identities.

(a) $\sin(A)\cos(B) = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$

(b) $\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

(c) $\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$.

$7x = A \qquad 3x = B$

Example: $\displaystyle\int \cos(7x)\cos(3x)\,dx$

((Case (c))

$$= \frac{1}{2}\int \cos(7x-3x) + \cos(7x+3x)\,dx$$

$$= \frac{1}{2}\int \cos(4x) + \cos(10x)\,dx$$

$$= \frac{1}{2}\left(\frac{\sin(4x)}{4} + \frac{\sin(10x)}{10}\right) + C.$$

§ 7.3. Trig. Substitution.

Usual Substitution.

$$\int f(g(x))\,g'(x)\,dx = f(g(x))\,dg(x) = \int f(u)\,du.$$

let $u = g(x)$

Inverse Substitution

$$\int f(x)\,dx = \int f(g(t))\,dg(t)) = \int f(g(t))\,g'(t)\,dt.$$

$x = g(t)$

NOTE: to use inverse substitution

i.e. $x = g(t)$

ONLY possible if g is 1–1 and onto

so that g has an inverse.

Example. $\int \sqrt{16 - x^2}\, dx$

let $x = 4\sin\theta = g(\theta)$ (then $g(\theta)$ needs to have an inverse.).