§7.1 cont'd Integration by Parts.

Example. $I = \int \ln(x)\,dx = \int \underbrace{(\ln x)}_{u} \underbrace{(1)\,dx}_{dv}$

Let $u = \ln(x)$ $\qquad v = x \qquad v = \int 1\,dx$

$du = \frac{1}{x}\,dx \qquad dv = 1\,dx$

$\int u\,dv = uv - \int v\,du$

$I = \underset{v}{x}\,\underset{u}{\ln x} - \int \underset{v}{x} \underbrace{\left(\frac{1}{x}\right) dx}_{du}$

$= x \ln x - \int 1\,dx = x\ln x - x + C.$

(check by differentiating)

Example. $I = \int \arctan(x)\,dx = \int \underbrace{\arctan(x)}_{u} \cdot \underbrace{1\,dx}_{dv}$

$u = \arctan(x) \qquad v = x$

$du = \frac{1}{1+x^2}\,dx \qquad dv = 1 \cdot dx$

$$I = x\arctan(x) - \int \frac{x}{1+x^2}\,dx$$

Now use Substitution Rule.

Let $u = 1+x^2$

$du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$

$$= x\arctan(x) - \left[\frac{1}{2}\int \frac{1}{u}\,du\right]$$

$$= x\arctan(x) - \frac{1}{2}\ln|u| + C$$

$$= x\arctan(x) - \frac{1}{2}\ln|1+x^2| + C$$

We used both Integration by parts & then Substitution Rule.

Choosing u and v in integration by parts

$$\int \underbrace{f(x)}_{u}\,\underbrace{g'(x)\,dx}_{dv}$$

$u = f(x)$

$dv = g'(x)\,dx$

Want to differentiate $f(x)$ (a u) to make it simpler (or at least not worse)

Want to integrate $g'(x)$

Best u's : $\ln x$, $\tan^{-1}(x)$, $\sin^{-1}(x)$

Good if differentiating. (not so good for $\int$ing.)

Also good for u : $x, x^2, \ldots x^n$

but you might have to use $\int$ by parts several times if $n>1$.

Not so good (most of the time) for u : e^x, $\cos(x)$, $\sin(x)$, $\cosh(x)$, $\sinh(x)$

(NOT ALWAYS as in next example).

Example. $I = \int \underbrace{e^x}_{u} \underbrace{\sin(x)\,dx}_{dv}$

$u = e^x$ $\qquad$ $v = -\cos(x)$

$du = e^x\,dx$ $\qquad$ $dv = \sin(x)\,dx$

$$I = -e^x\cos(x) - \int(-\cos(x))e^x\,dx$$

$u = \cos(x)$

$du = -\sin(x)$

$v = e^x$

$dv = e^x\,dx$

Repeat.

$$= -e^x\cos(x) + \int \underbrace{\cos(x)}_{u}\underbrace{e^x\,dx}_{dv}$$

$$= -e^x\cancel{\cos(x)} + \left\{\cancel{e^x\cos(x)} + \int e^x\sin(x)\,dx\right.$$

= back to the start.

(Just undid what we did first.)

Not helpful!

Instead. $u = e^x$ $\quad$ $v = \sin(x)$

$du = e^x\,dx$ $\quad$ $dv = \cos(x)\,dx$

(Use the same thing for u the 2nd time.)

$$\int e^x\sin(x)\,dx = -e^x\cos(x) + \left\{e^x\sin(x) - \int e^x\sin(x)\,dx\right\}$$

$$2\int e^x\sin(x)\,dx = -e^x\cos(x) + e^x\sin(x) + C.$$

$$\int e^x\sin(x)\,dx = \frac{1}{2}\left(-e^x\cos(x) + e^x\sin(x)\right) + C.$$

Example: $I = \int_{x=1}^{4} e^{\sqrt{x}}\, dx$

Combine substitution Integration by parts FTh Cal PII.

Sol'n. Let $w = \sqrt{x} = x^{\frac{1}{2}}$

$dw = \frac{1}{2} x^{-1/2} dx \Rightarrow \boxed{dx = 2x^{1/2} dw = 2w\,dw}$

$w = \sqrt{x}$

$x=1 \Rightarrow w=1$
$x=4 \Rightarrow w=2$

Substitution so must change the limits of integration.

$$I = \int_{w=1}^{2} 2e^{w} w\, dw$$

Now we use integration by parts.

$u = w \qquad v = 2e^{w}$

$du = dw \qquad dv = 2e^{w} dw$

This is NOT a substitution, so it does NOT affect the limits of integration.

$$I = 2we^{w}\Big|_{w=1}^{2} - \int_{1}^{2} 2e^{w} dw$$

$$= \left((2)(2e^{2}) - 2e\right) - \left(2e^{w}\Big|_{1}^{2}\right)$$

$$= 4e^2 - \cancel{2e} - (2e^2 - \cancel{2e})$$

$$= 2e^2$$

Example (Example 6 in Textbook pg 475)

Reduction formula.

$I = \int \sin^n(x)\,dx$ n integer, $n \geq 2$.

$$= -\frac{1}{n}\cos x \sin^{n-1}(x)\,dx + \frac{n-1}{n}\int \sin^{n-2} x \, dx$$

Note: $\sin^n(x) = \sin^{n-1}(x)\sin(x)$.

$$I = \int \overbrace{\sin^{n-1}(x)}^{u}\ \underbrace{\overbrace{\sin(x)}^{dv}\,dx}.$$

$u = \sin^{(n-1)}(x)$ $v = -\cos(x)$

$du = (n-1)\sin^{n-2}(x)\cos(x)\,dx$ $dv = \sin(x)\,dx$

$$= -\cos(x)\sin^{n-1}(x) - \int -\cos(x)\,(n-1)\sin^{n-2}(x)\cos x\,dx$$

$$= -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\cos^2(x)\,dx.$$

$$\cos^2 x = 1 - \sin^2(x)$$

$$= -\cos x \sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\left(1-\sin^2 x\right)dx$$

$$\int \sin^n(x)\,dx = -\cos x \sin^{n-1}(x) + (n-1)\left(\int \sin^{n-2}(x)\,dx - \int \sin^n(x)\,dx\right)$$

$$n\int \sin^n(x)\,dx = \left(-\cos x \sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\,dx\right)$$

$$\therefore \int \sin^n(x)\,dx = \frac{1}{n}\left(-\cos x \sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\,dx\right)$$

$n \geq 2$, integer.

Using formula:

$n=2$

$$\int \sin^2(x)\,dx = \frac{1}{2}\left(-\cos(x)\sin x + \int 1\,dx\right) + C$$

$$= \frac{1}{2}\left(-\cos(x)\sin(x) + x\right) + C.$$

$n=3$. $\int \sin^3(x)\,dx = \frac{1}{3}\left(-\cos(x)\sin^2(x) + 2\int \sin(x)\,dx\right)$.

$= \frac{1}{3}\left(-\cos(x)\sin^2(x) - 2\cos(x)\right) + C$.

Try $n=4$ this way (i.e. using the reduction formula.

$n=4$. $I = \int \sin^4 x\,dx = \frac{1}{4}\left(-\sin x\cos^3 x + 3\int \sin^2(x)\,dx\right)$

Use formula again:

$I = \frac{1}{4}\left[-\cos x\sin^3 x + 3\left\{\frac{1}{2}\left(-\cos x\sin x + x\right)\right\}\right] + C$

See Example 4, page 480 of §7.2 where the solution looks different, but using trig identities, we can show they are equal.

$I = \frac{1}{4}\left[\frac{3}{2}x - \sin x\cos x\left(\frac{3}{2} + \sin^2 x\right)\right] + C$

$= \frac{1}{4}\left[\frac{3}{2}x - \frac{3}{4}\sin(2x) - \frac{1}{2}\sin(2x)\sin^2 x\right] + C$

Using $\sin x\cos x = \frac{1}{2}\sin 2x$.

$$= \frac{1}{4}\left[\frac{3}{2}x - \frac{3}{4}\sin(2x) - \frac{1}{2}\sin(2x)\left(\frac{1}{2}(1-\cos(2x)\right)\right. + C$$

Using $\sin^2 x = \frac{1}{2}(1-\cos(2x))$

$$= \frac{1}{4}\left[\frac{3}{2}x - \sin(2x) + \frac{1}{4}\sin(2x)\cos(2x)\right] + C$$

$$= \frac{1}{4}\left[\frac{3}{2}x - \sin(2x) + \frac{1}{8}\sin(4x)\right] + C$$

(The solution in the book.)