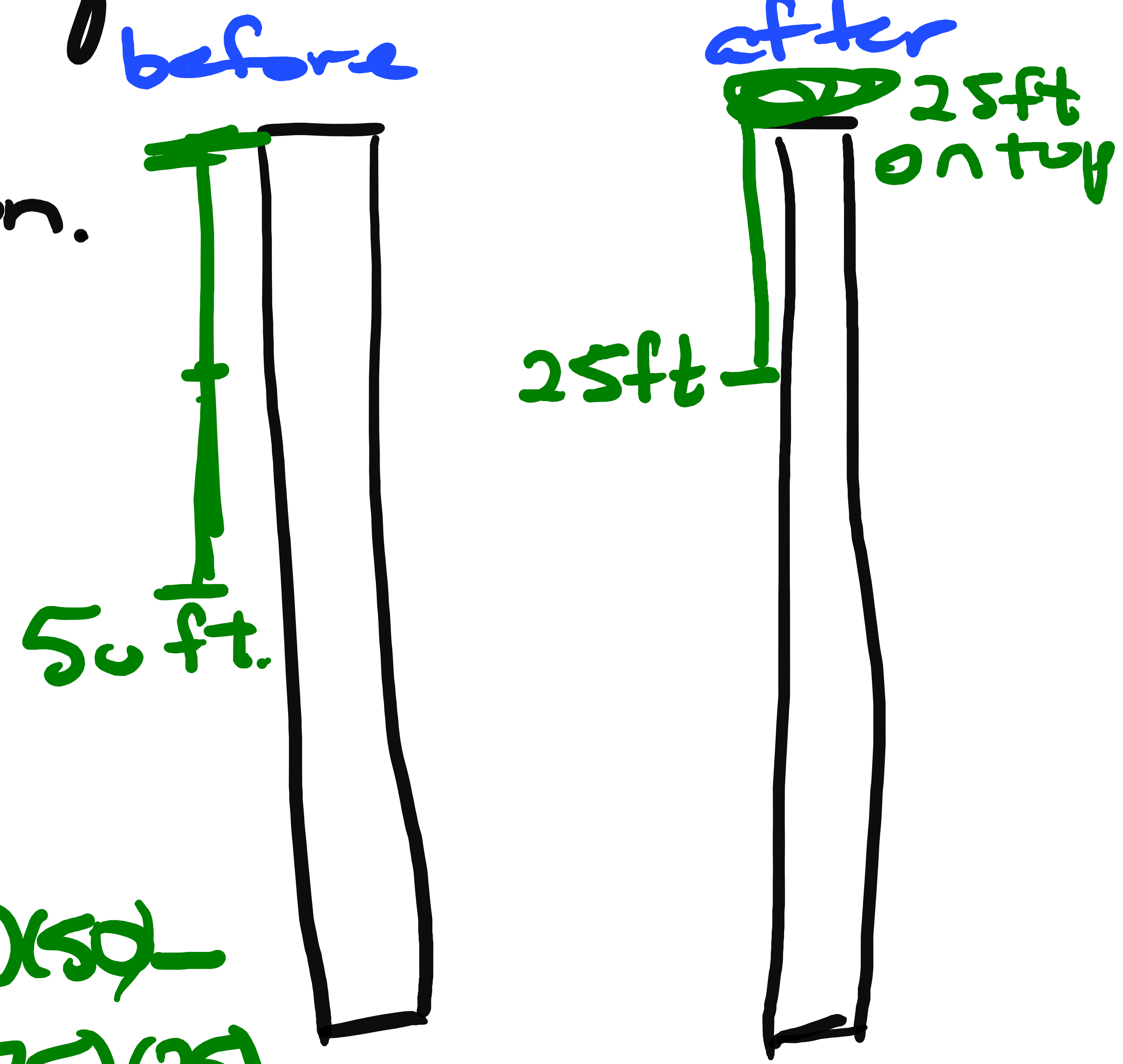
Cont'd from last lecture.

We still need to find the work required work to pull the bottom 1/2 of the chain up so that only half of it still hangs down.



Every point on the bottom half of the chain is pulled up 25ft.

$$W_2 = \int_{25}^{50} (3)\underbrace{(25)}_{\text{ft}}\,dt = 75t\Big|_{t=25}^{50} = (75)(50) - (75)(25)$$

lb/ft

ft

Every point on the chain below the original 25ft gets pulled up exactly 25ft.

The total amount of work required to pull the chain up so that the top half of the chain lies on the top of the building is $W_1 + W_2$.

$$= \left(\overbrace{\frac{3(25)^2}{2}}^{\text{top half}} + \overbrace{75(25)}^{\text{bottom half.}}\right) \ \underline{\text{ft-lb.}}$$

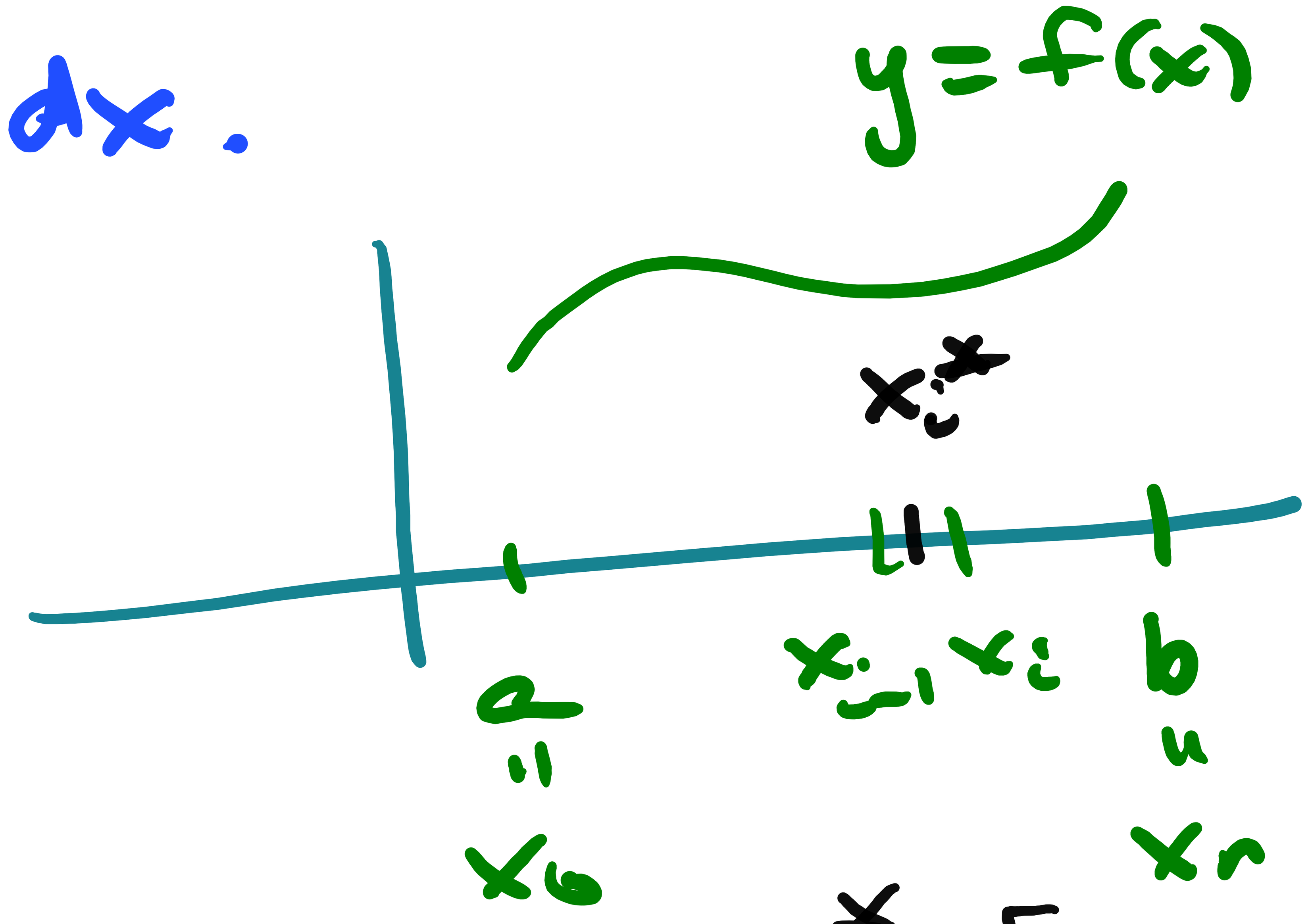
§6.5 Average Value of a Function.

$\{n_1, n_2, \ldots, n_n\}$, n numbers.

Their AVERAGE is : $\frac{1}{n}\sum_{i=1}^{n} n_i = \frac{1}{n}(n_1 + n_2 + \cdots + n_n)$.

If $y = f(x)$. The AVERAGE VALUE of f on $[a,b]$ is :

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$



$x_i^* \in [x_{i-1}, x_i]$

<u>Why</u>? Usual IDEA. PARTITION.

$\Delta x = \frac{b-a}{n}$ (equal subintervals)

Approx
Avg height $\approx \frac{1}{n}\sum_{i=1}^{n} f(x_i^*)$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(f(x_i^*)\,\frac{(b-a)}{(b-a)}\right)$$

$$= \frac{1}{(b-a)} \sum_{i=1}^{n} f(x_i^*)\left(\frac{(b-a)}{n}\right) \qquad \left(\Delta x = \frac{b-a}{n}\right).$$

Let $n \to \infty$ to get the actual AVERAGE.

$$\lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*) \underbrace{\frac{(b-a)}{n}}_{\Delta x} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example. Find the AVERAGE VALUE of $y = x^3$ for $x \in [1, 5]$.

Sol'n. $$y_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\underset{b}{5} - \underset{a}{1}}\int_1^5 \underbrace{x^3}_{f(x)}\,dx$$

$$= \frac{1}{4}\left(\frac{x^4}{4}\right)\Bigg|_1^5$$

$$= \frac{1}{4}\left(\frac{5^4}{4} - \frac{1}{4}\right) = \frac{1}{16}(624).$$

Example In a certain city the temperature in °C t hours after 11 AM was given by

$$f(t) = 10 - 10\sin\left(\frac{\pi t}{12}\right).$$

t hours.

Find the AVERAGE temperature during the period from 11 AM until 5PM.

Sol'n: Let $t=0$ denote 11 AM. (t in hours)

Then 5 PM is 6 hrs. later, i.e. $t=6$.

$$\text{AVG TEMP} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{\underset{b}{6} - \underset{a}{0}}\int_0^6 \underbrace{10 - 10\sin\left(\frac{\pi t}{12}\right)}_{f(t)}\,dt$$

$$= \frac{1}{6}\left\{\left(10t + 10\cos\left(\frac{\pi t}{12}\right)\left(\frac{12}{\pi}\right)\right|_0^6\right.$$

$$= \frac{1}{6}\left[\left(10(6) + 10\cos\left(\frac{\pi}{2}\right)\left(\frac{12}{\pi}\right)\right) - \left(0 + \frac{10(12)}{\pi}\right)\right]$$

$= 10 + 0 - 20/\pi = \underline{10 - 20/\pi}\ ^\circ C.$ AVG. TEMP.

Mean Value Thm for Integrals.

f continuous on $[a,b]$.

There exists $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Pf. Let $F(x) = \int_a^x f(t)\,dt$, $x \in [a,b]$.

$\therefore F(x)$ is continuous on $[a,b]$ and differentiable on (a,b).

$\therefore$ by the Mean Value Thm, there exists $c \in [a,b]$ such that

$$F(b) - F(a) = F'(c)(b-a).$$

But $F'(x) = f(x)$ F Th Calc. P1.

$$\therefore F(b) - F(a) = \int_a^b f(t)\,dt - \int_a^a \cancel{f(t)}\,dt = f(c)(b-a)$$

$$\Rightarrow \int_a^b f(t)\,dt = f(c)(b-a)$$

$$\Rightarrow f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

§7.1 Integration by Parts.

(related to PRODUCT RULE for differentiation.)

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

Integrate both sides

$$\Rightarrow \boxed{\int \left(f'(x)g(x) + f(x)g'(x)\right)dx = f(x)g(x)}$$

Rearrange $\Rightarrow$ $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$ ($*$)

Let $u = f(x)$ $\qquad v = g(x)$

$du = f'(x)\,dx$ $\qquad dv = g'(x)\,dx$

By substituting into ($*$)

$$\int \underbrace{f(x)}_{u}\underbrace{g'(x)\,dx}_{dv} = \underbrace{f(x)}_{u}\underbrace{g(x)}_{v} - \int \underbrace{f'(x)\,dx}_{du}\,\underbrace{g(x)}_{v}$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Integration by parts.

Example: $I = \int \underbrace{x}_{u} \underbrace{e^x\,dx}_{dv}$

$u = x \qquad v = e^x$

$du = dx \qquad dv = e^x\,dx$

$$I = \int u\,dv = uv - \int v\,du$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C$$

Check by differentiating.

Example: Sometimes you need to repeat more than once.

$$I = \int \underbrace{x^2}_{u} \underbrace{\sin(x)\,dx}_{dv}$$

let $u = x^2$ $\qquad v = -\cos(x)$

$du = 2x\,dx$ $\qquad dv = \sin(x)\,dx$

$$= \underset{u}{x^2}\underbrace{(-\cos(x))}_{v} - \int \underbrace{(-\cos(x))}_{v}\,\underbrace{2x\,dx}_{du}$$

$$= -x^2\cos(x) + 2\int \underset{u}{x}\,\underbrace{\cos(x)\,dx}_{dv}$$

Again: $\qquad u = x \qquad v = \sin(x)$

$\qquad du = dx \qquad dv = \cos(x)\,dx$

$$= -x^2\cos(x) + 2\left\{\underset{u}{x}\,\underset{v}{\sin(x)} - \int \underset{v}{\sin(x)}\,\underset{du}{dx}\right\}$$

$$= -x^2\cos(x) + 2\left\{x\sin(x) + \cos(x)\right\} + C.$$

Check by differentiating.