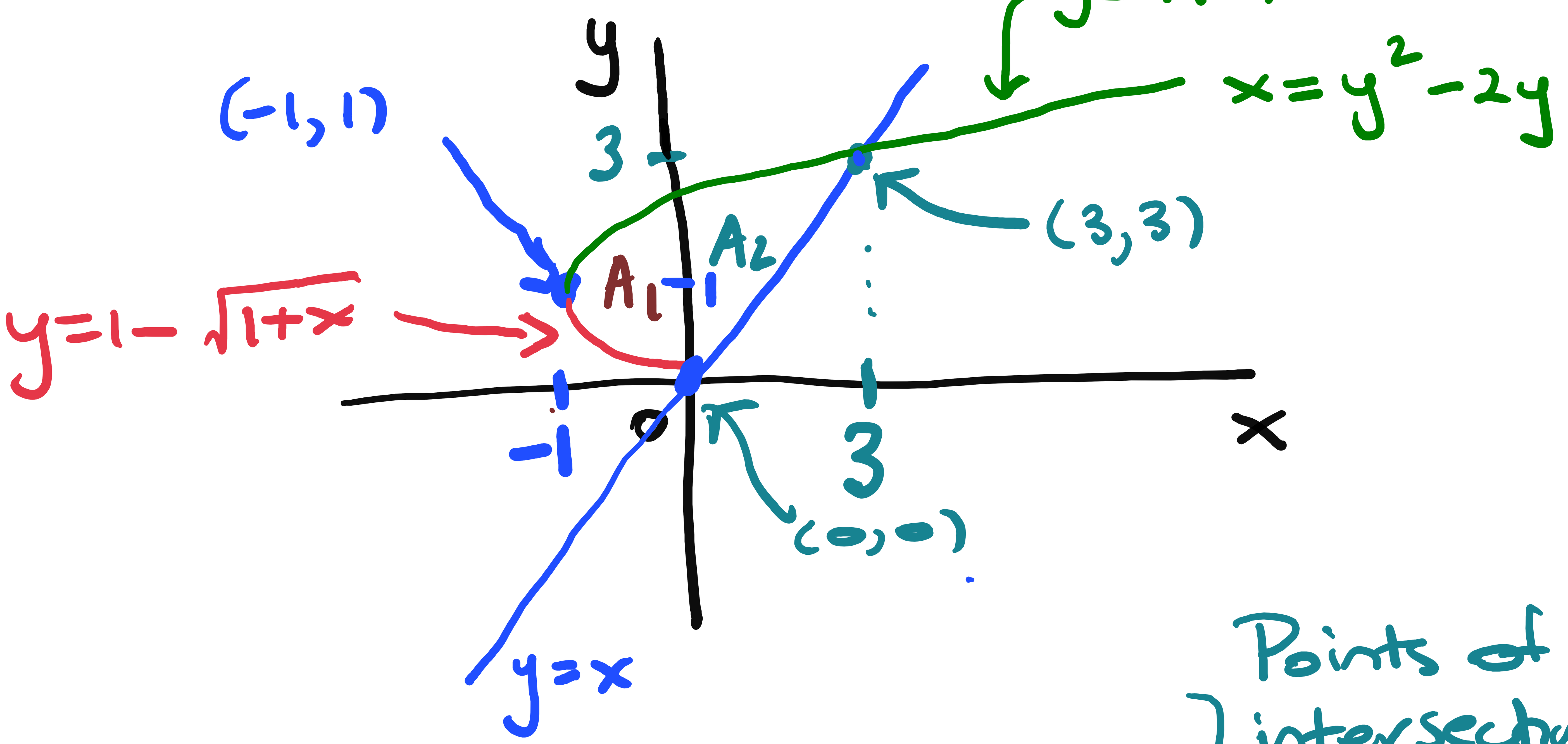
M1A03 Calculus I Section C01 Dr. Wolkowicz Lecture 27 November 7, 2019

Information about Test 2 posted on the course Web-site as for test 1

§6.1 cont'd.

Example. Find the area of the region bounded by the parabola $x=y^2-2y$ and the line $y=x$.



Sol'n. Draw graph. Find points of intersection:

Solve: $y^2-2y=y$

$\Rightarrow y^2-3y=0$

$y(y-3)=0$

$y=0$ & $y=3$.

If $y=0 \Rightarrow x=0$. If $y=3$: $x=y^2-2y=(3)^2-2(3)=3$.

Points of intersection are $(0,0)$ & $(3,3)$ (Check!).

Find the left most point on the parabola:

$$x=y^2-2y=f(y), \qquad f'(y)=2y-2=0 \Rightarrow y=1$$

$y=1$ corresponds to $x = y^2 - 2y = (1)^2 - 2(1) = -1$.

Area of Region $= A_1 + A_2$. We need equations of upper curve and lower curve where $y = y(x)$.

Solve for y as a function of x:

$$x = y^2 - 2y \Rightarrow y^2 - 2y - x = 0$$

From quadratic formula: $y = \dfrac{2 \pm \sqrt{4 - 4(-x)}}{2} = 1 \pm \sqrt{1+x}$

(Note: $\dfrac{\sqrt{4}}{2} = \sqrt{\dfrac{4}{4}}$)

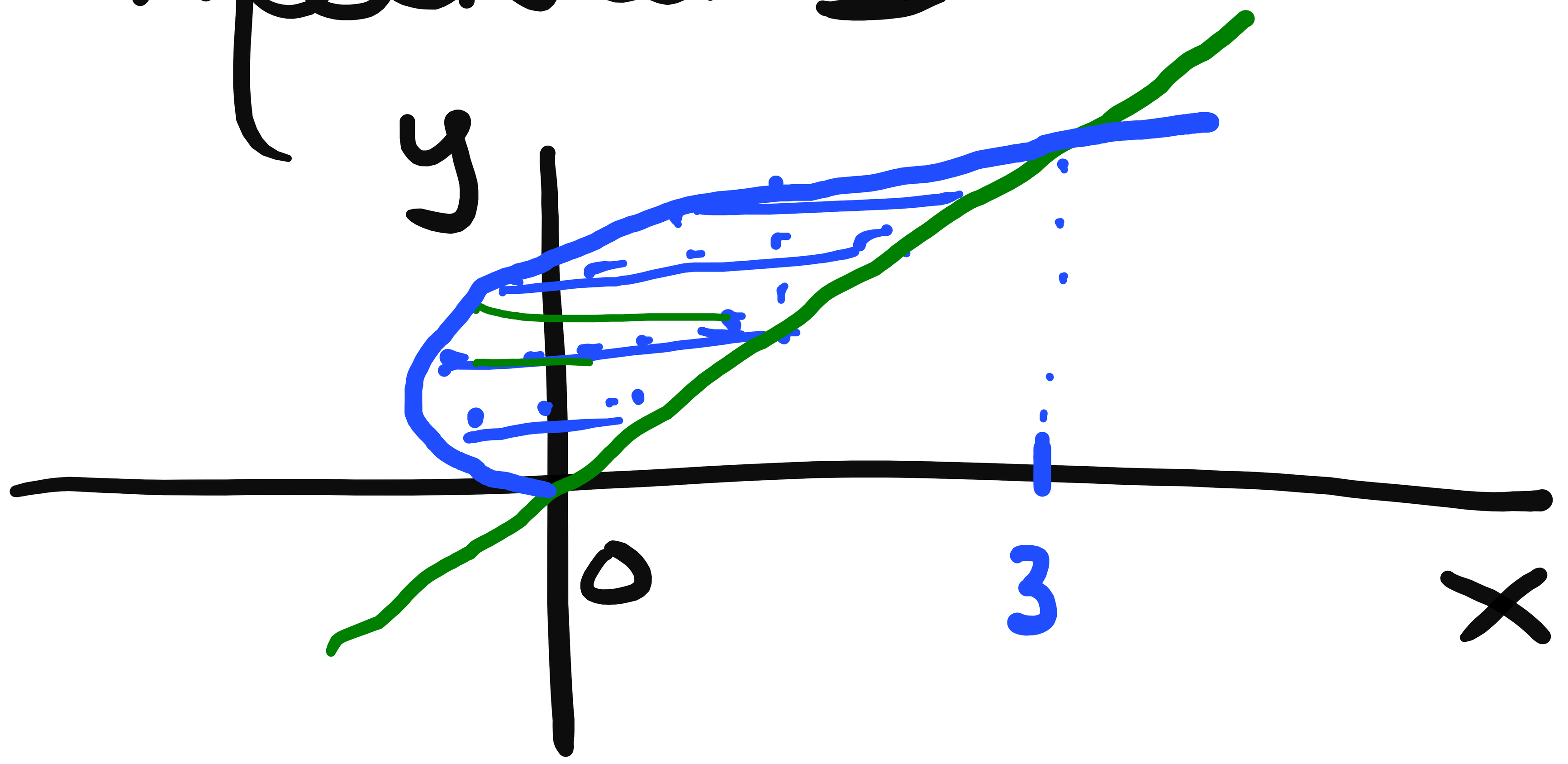
$y(x) = 1 + \sqrt{1+x}$ (top)

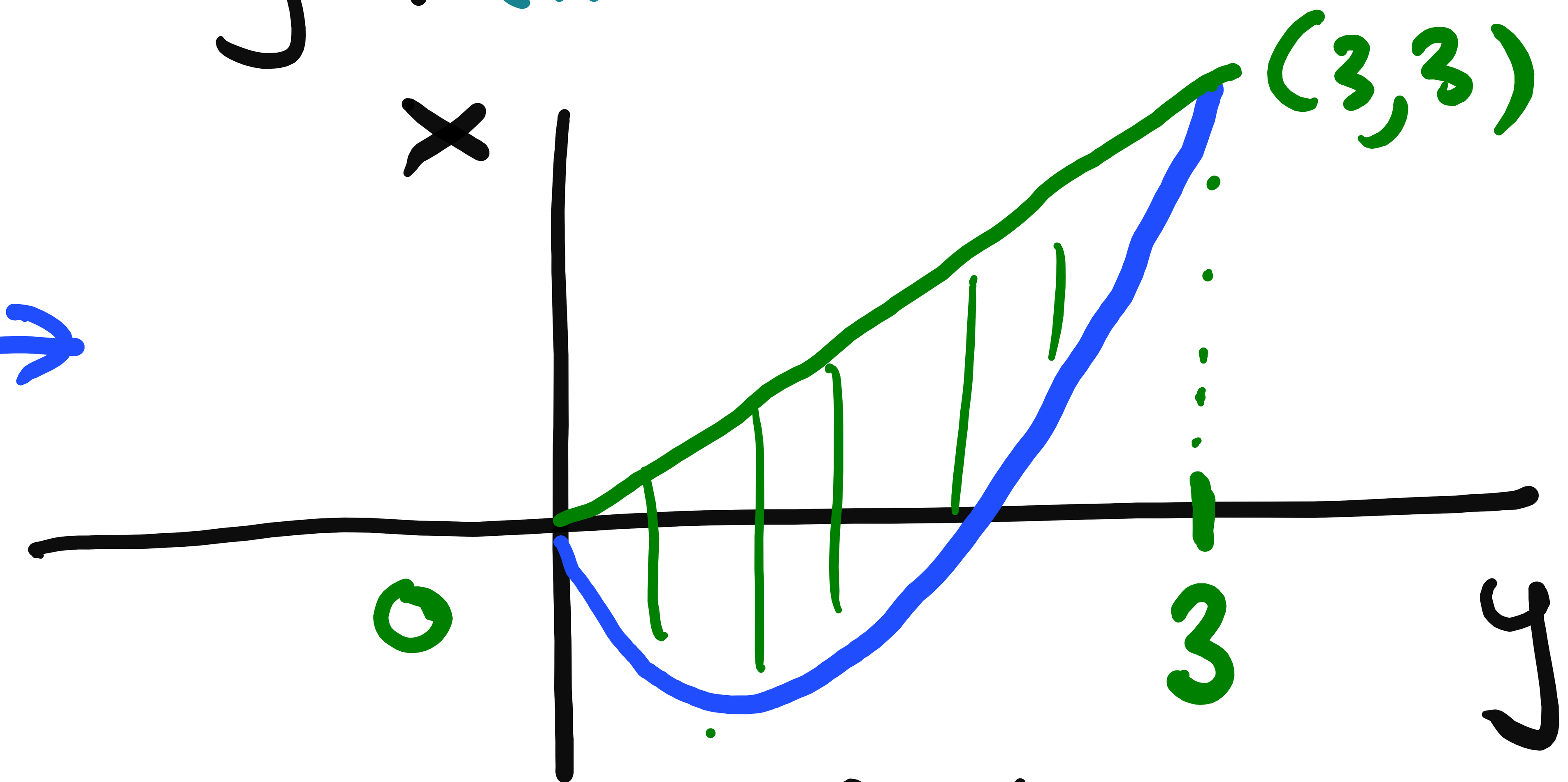
$y(x) = 1 - \sqrt{1+x}$ (bottom)

$$A_1 + A_2 = \int_{-1}^{0} \underbrace{(1 + \sqrt{1+x})}_{\text{top}} - \underbrace{(1 - \sqrt{1+x})}_{\text{bottom}}\, dx$$

$$+ \int_0^3 \underbrace{(1 + \sqrt{1+x})}_{\text{(top)}} - \underbrace{(x)}_{\text{(bottom)}}\, dx$$

Method II (An easier way). (Reverse roles of x and y.)





Take strips from right to left instead of top to bottom.

$$A = \int_0^3 (y) - (y^2 - 2y)\, dy$$

$$= \left(\frac{y^2}{2} - \left(\frac{y^3}{3} - \frac{2y^2}{2}\right)\right)\Big|_0^3 = \left(\frac{9}{2} - \left(\frac{27}{3} - 9\right)\right)$$

$$= \frac{9}{2}$$

(Note: $\frac{27}{2} - 9 = \frac{27}{2} - \frac{18}{2} = \frac{9}{2}$).

Integrating using the first method:

$$\int \sqrt{1+x}\, dx = \int u^{1/2}\, du$$

using substitution.

$u = 1+x$

$du = dx$

$$= 2\frac{u^{3/2}}{3}$$

$$= \frac{2}{3}(1+x)^{3/2}$$

$$A_1 + A_2 = \int_{-1}^{0} 2\sqrt{1+x}\, dx + \int_0^3 1 + \sqrt{1+x} - x\, dx$$

$$= 2\left(\frac{2}{3}\right)(1+x)^{3/2}\Big|_{-1}^{0} + \left(x + \frac{2}{3}(1+x)^{3/2} - \frac{x^2}{2}\right)\Big|_0^3$$

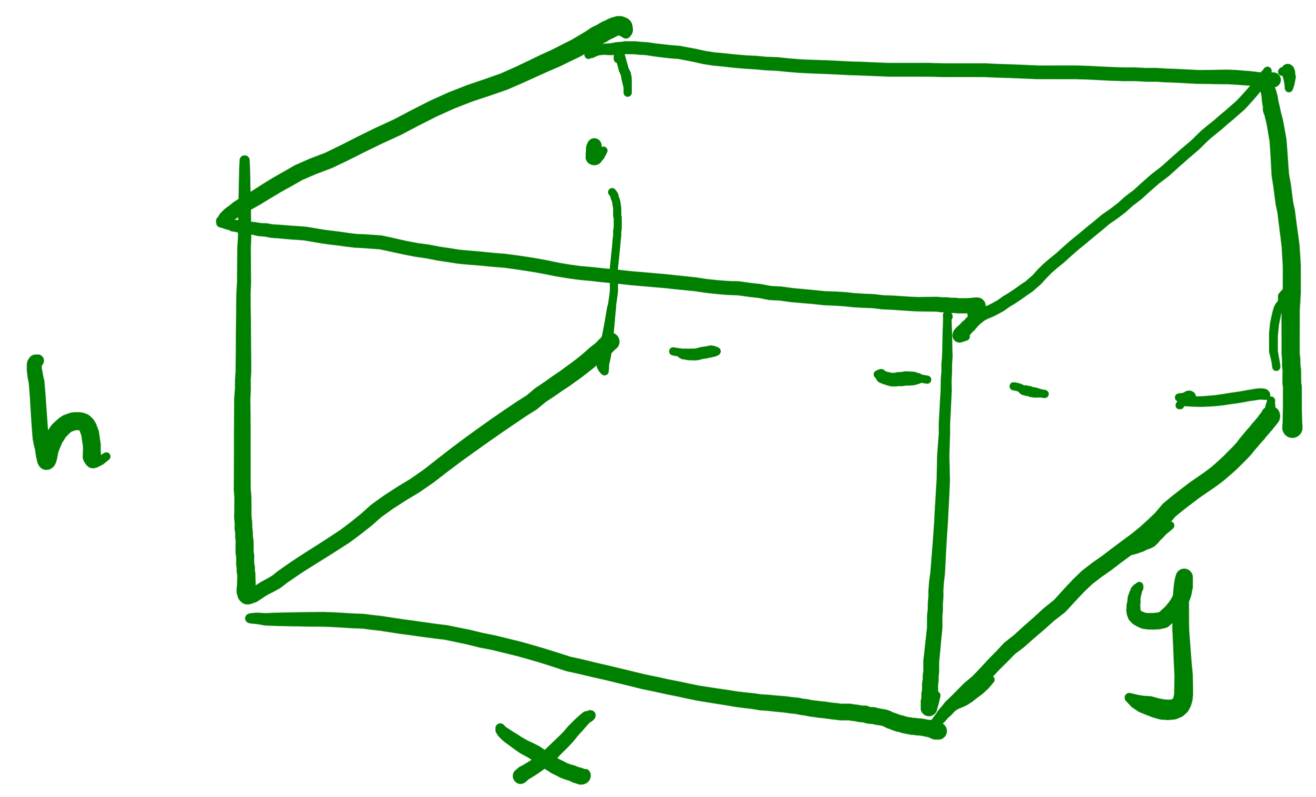
$$= \frac{4}{3}(1-0) + \left(3 + \frac{2}{3}(4)^{3/2} - \frac{9}{2}\right) - \left(\frac{2}{3}\right)$$
$$= \frac{9}{2}.$$

§6.2 Volumes.

V used for volume
A used for area.

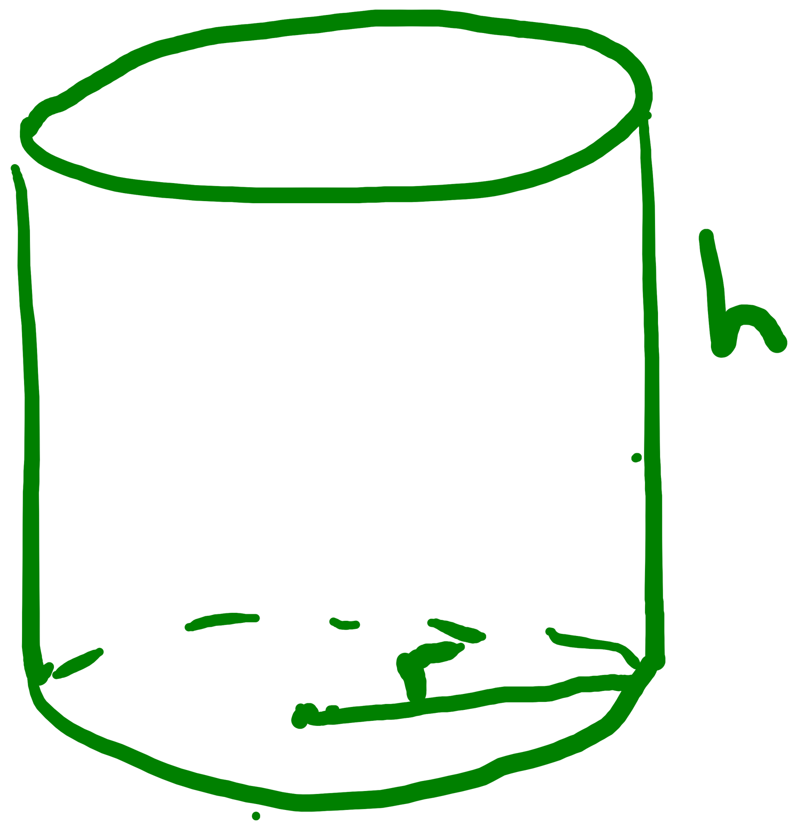
Examples of RIGHT Cylinders.

1. Rectangular box volume.



$$\text{Volume} = \underbrace{(xy)}_{\text{area of the base}} \cdot \underbrace{h}_{\text{height}}.$$

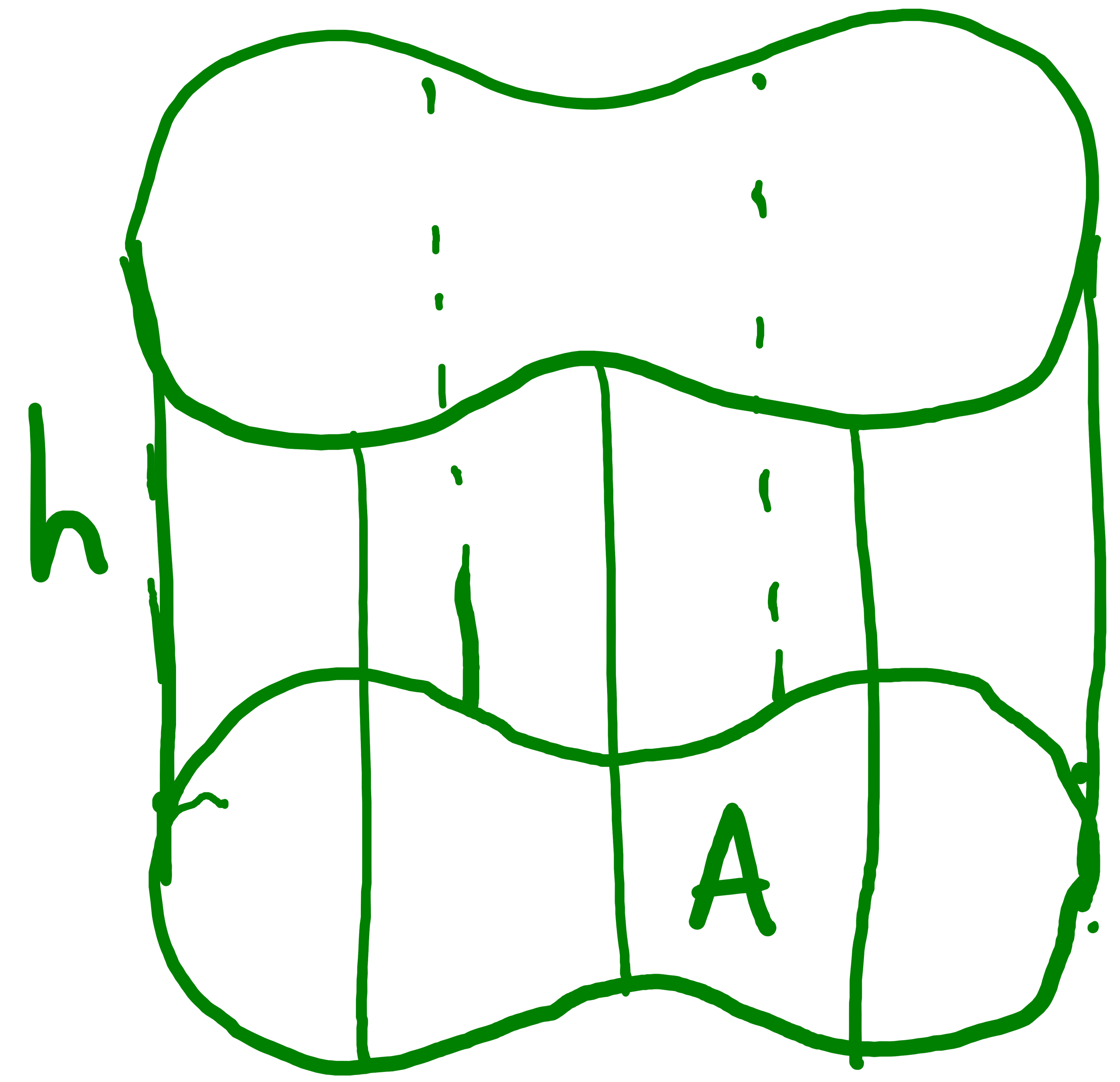
2. Right CIRCULAR cylinder.



(radius r)

$$V = \underbrace{(\pi r^2)}_{\text{area of the base}} \underbrace{h}_{\text{height}}$$

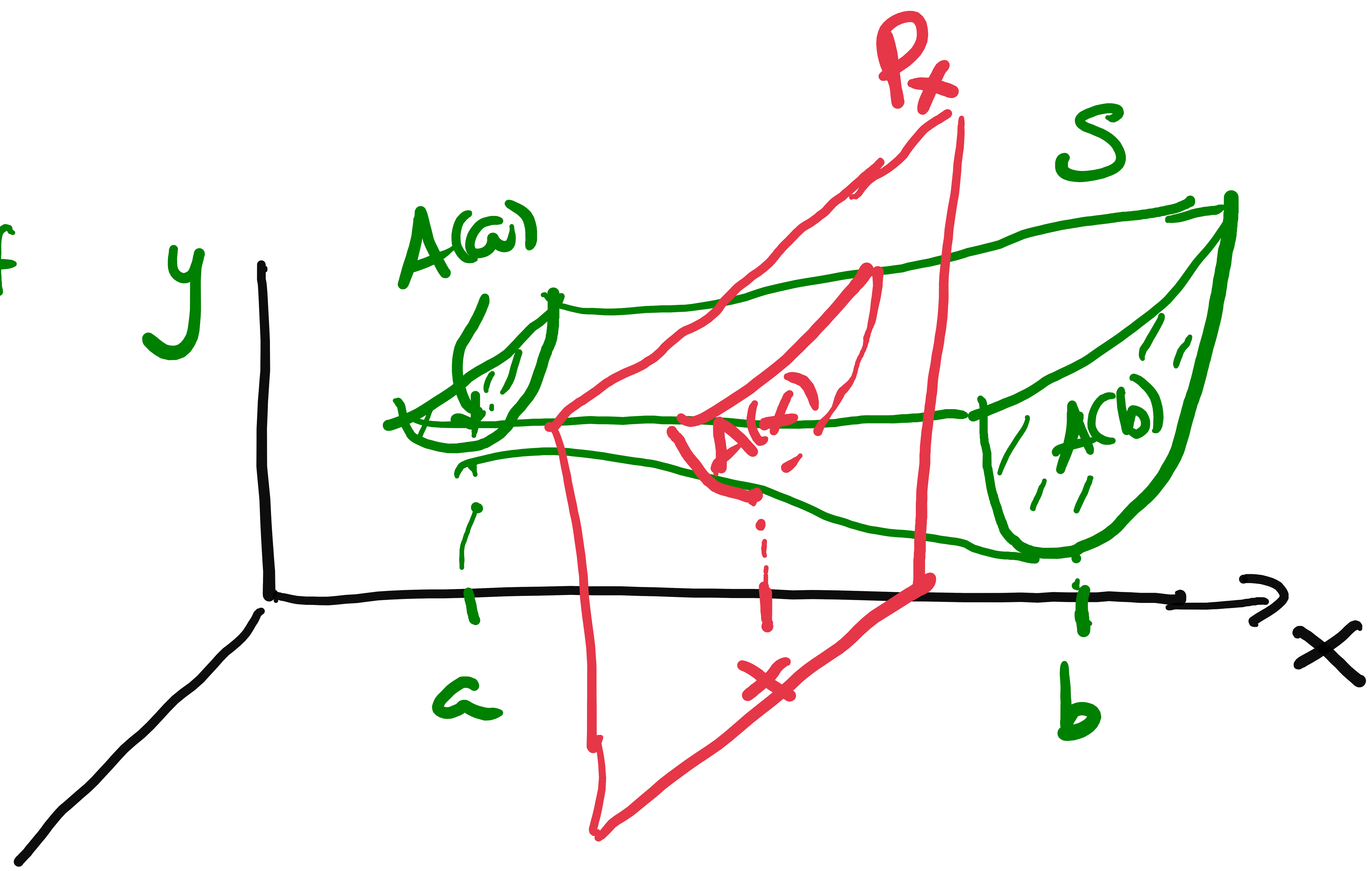
3. More generally, volume of any right cylinder with area of the base A and height h:



$$V = \underbrace{A}_{\text{area of the base}} \; \underbrace{h}_{\text{height}}.$$

VOLUME of ANY SOLID.

Let $A(x)$ denote the area of the cross-section of S in the plane P_x through x perpendicular to the x-axis for $x \in [a,b]$



ε

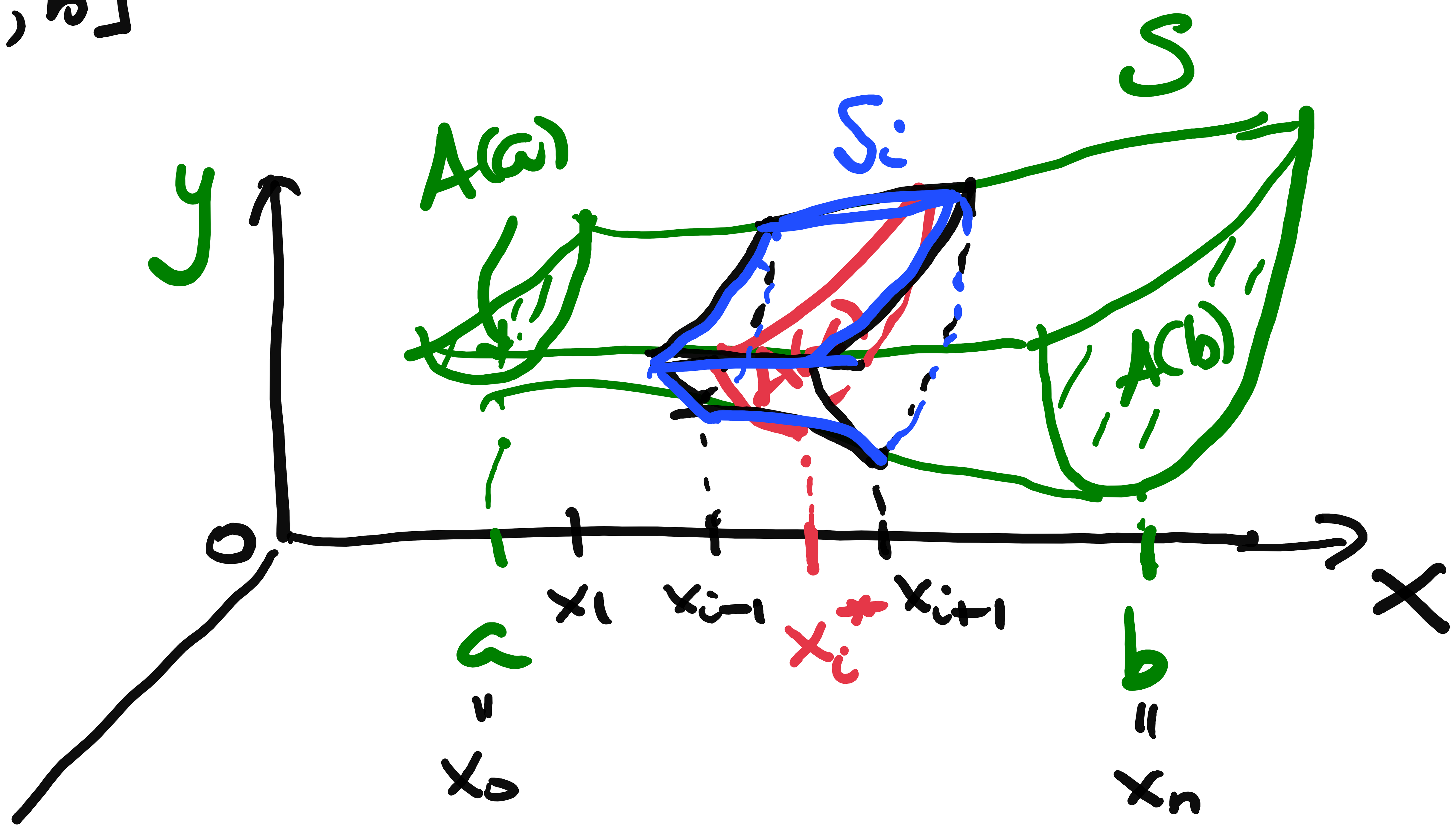
Let P be a partition of $[a,b]$

$a = x_0 < x_1 < x_2 < \cdots < x_n = b$

Choose $x_i^* \in [x_{i-1}, x_i]$

Let $\Delta x_i = x_i - x_{i-1}$

Let $\|P\| = \max\{\Delta x_i\}$.



Let S_i be the i^{th} slice
i.e., the part of S lying
between $P_{x_{i-1}}$ and P_{x_i}
the planes through x_{i-1} and x_i, respectively
perpendicular to the x-axis.

The volume of $S_i \approx A(x_i^*)\Delta x_i$ (i.e. approximate volume of S_i as if it was a right cylinder).

approximate total volume of $S \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i$

approximation improves as $\|P\| \to 0$.
i.e. as slices get thinner $(n \to \infty)$.

Define the VOLUME of S as

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x_i^*) \Delta x_i$$ (if the limit exists)

$$\therefore \; V = \int_a^b \underbrace{A(x)}_{\text{moving cross-sections}} dx$$ (by the def'n of the definite integral).

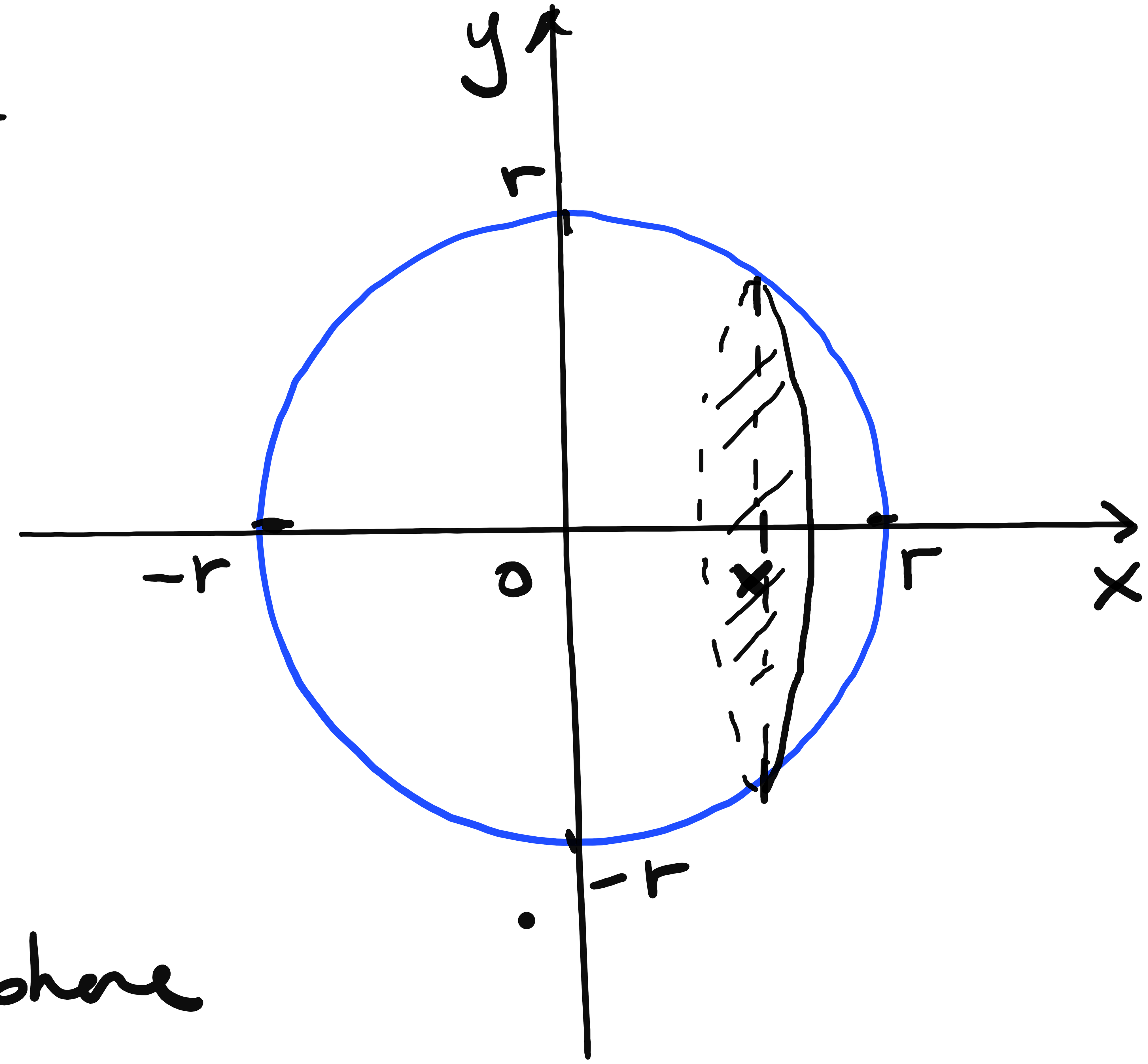
Example: Show the volume of a SPHERE is $V = \frac{4}{3}\pi r^3$.

Sol'n. Place the SPHERE with centre at the origin.

The plane P_x through x intersects the sphere in a circle.

To find its radius and hence its area:

$$x^2+y^2=r^2$$
$$y^2=r^2-x^2$$
$$y=\underbrace{\sqrt{r^2-x^2}}_{\text{(a radius is always positive)}}$$



$\therefore$ The plane P_x intersects the sphere in a circle with radius

$$y=\sqrt{r^2-x^2}$$

and has

Cross-sectional area $A(x)=\pi y^2=\pi(r^2-x^2)$

$$\underset{\text{volume}}{V}=\int_{-r}^{r}A(x)\,dx=\int_{-r}^{r}\pi(r^2-x^2)\,dx$$

$$=2\pi\int_0^r(r^2-x^2)\,dx$$

The integrand is an EVEN function & the interval is symmetric

$$= 2\pi\left(r^2x - \frac{x^3}{3}\right|_0^r$$ (r is a constant)

$$= 2\pi\left(\left(r^3 - \frac{r^3}{3}\right) - (0-0)\right)$$

$$= 2\pi\left(\frac{2r^3}{3}\right) = \frac{4\pi r^3}{3}.$$

Volume of a sphere with radius r.