Information about Test 2 posted on the course Web-site as for test 1

§ 5.5. cont'd. (Substitution Rule).

Definite $\int$ with substitution.

Method 1. Change the limits of integration using the change of variables

$$\int_{x=a}^{x=b} f'(\underbrace{g(x)}_{u}) \underbrace{g'(x)\,dx}_{du} = \int_{u=g(a)}^{u=g(b)} f'(u)\,du \qquad \begin{array}{l} b \to g(b) \\ a \to g(a) \end{array}$$

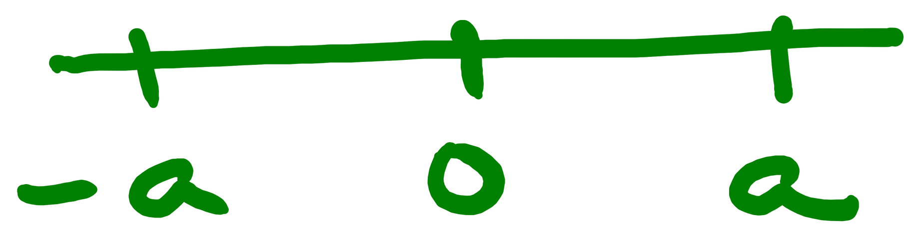
$u = g(x)$

$$\frac{du}{dx} = g'(x) \Rightarrow du = g'(x)\,dx$$

Method 2. Change the antiderivative back to the original variables (as for indefinite integrals) and then use the original limits of integration.

Taking advantage of SYMMETRY

Assume f is CONTINUOUS on $[-a,a]$.

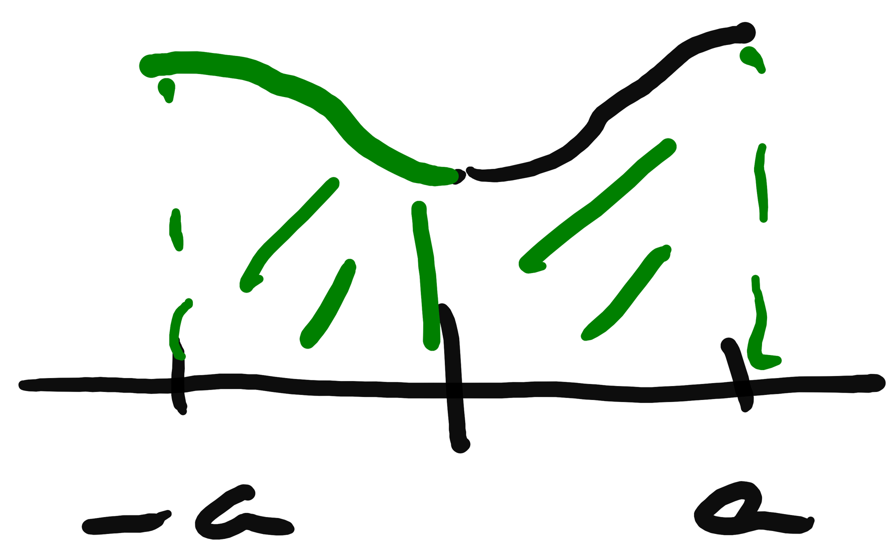


(a) If f is EVEN $(f(-x)=f(x))$

then $$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

Pf. $$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$$= -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx.$$



Use substitution in the 1st $\int$. i.e, $u=-x$, $du=-dx$, $x=0 \Rightarrow u=0$, $x=-a \Rightarrow u=a$

$$= -\int_0^a f(-u)(-du) + \int_0^a f(x)\,dx$$

$$= +\int_0^a \underbrace{f(-u)}_{f(u) \text{ since } f \text{ is EVEN}}(+du) + \int_0^a f(x)\,dx$$

$$= \int_0^a f(u)\,du + \int_0^a f(x)\,dx$$

$$= 2\int_0^a f(x)\,dx \qquad \left(\text{since } \int_0^a f(u)\,du = \int_0^a f(x)\,dx\right)$$

x is a dummy variable.

$$\int_0^a f(p)\,dp \text{ etc.}$$

"

(b) If f is ODD $(f(-x) = -f(x))$

then $\int_{-a}^{a} f(x)\,dx = 0$.

Pf. Same as in part (a), but instead of using $f(-u)=f(u)$, because f was even, use $f(-u)=-f(u)$, since f is ODD.

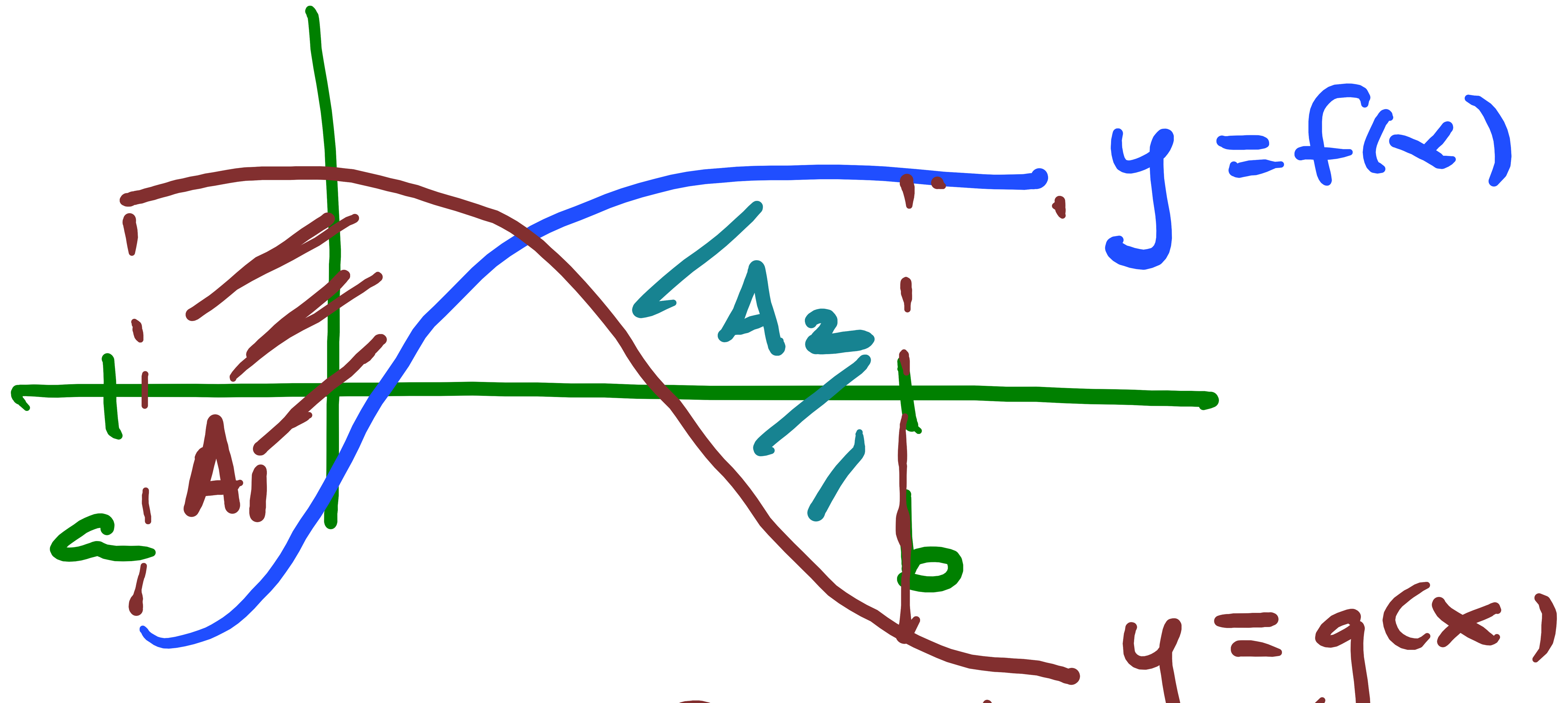
BEWARE:

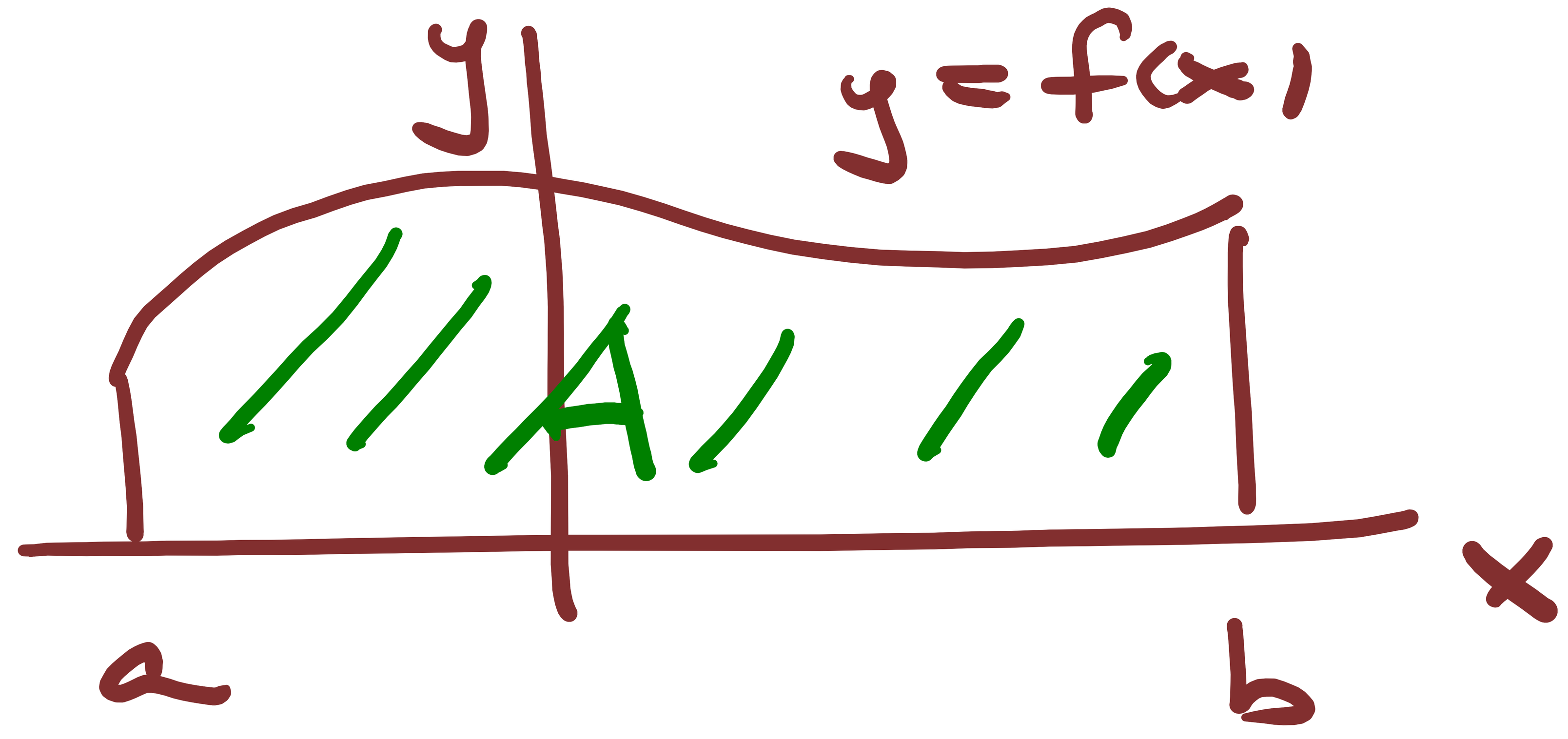
Example: $\int e^{x^2}\,dx$, substitution will not work. $f(x)=e^{x^2}$ does not have an anti-derivative in terms of "elementary functions" so this integral can only be done "numerically"; as a definite integral.

§6.1 AREAS between CURVES.

If $f(x) \geq 0$ & continuous $\left\{ \int_a^b f(x)\,dx \geq 0 \right.$ if $a \leq b$, and it gives the AREA under the curve from a to b above the x-axis.

Consider,

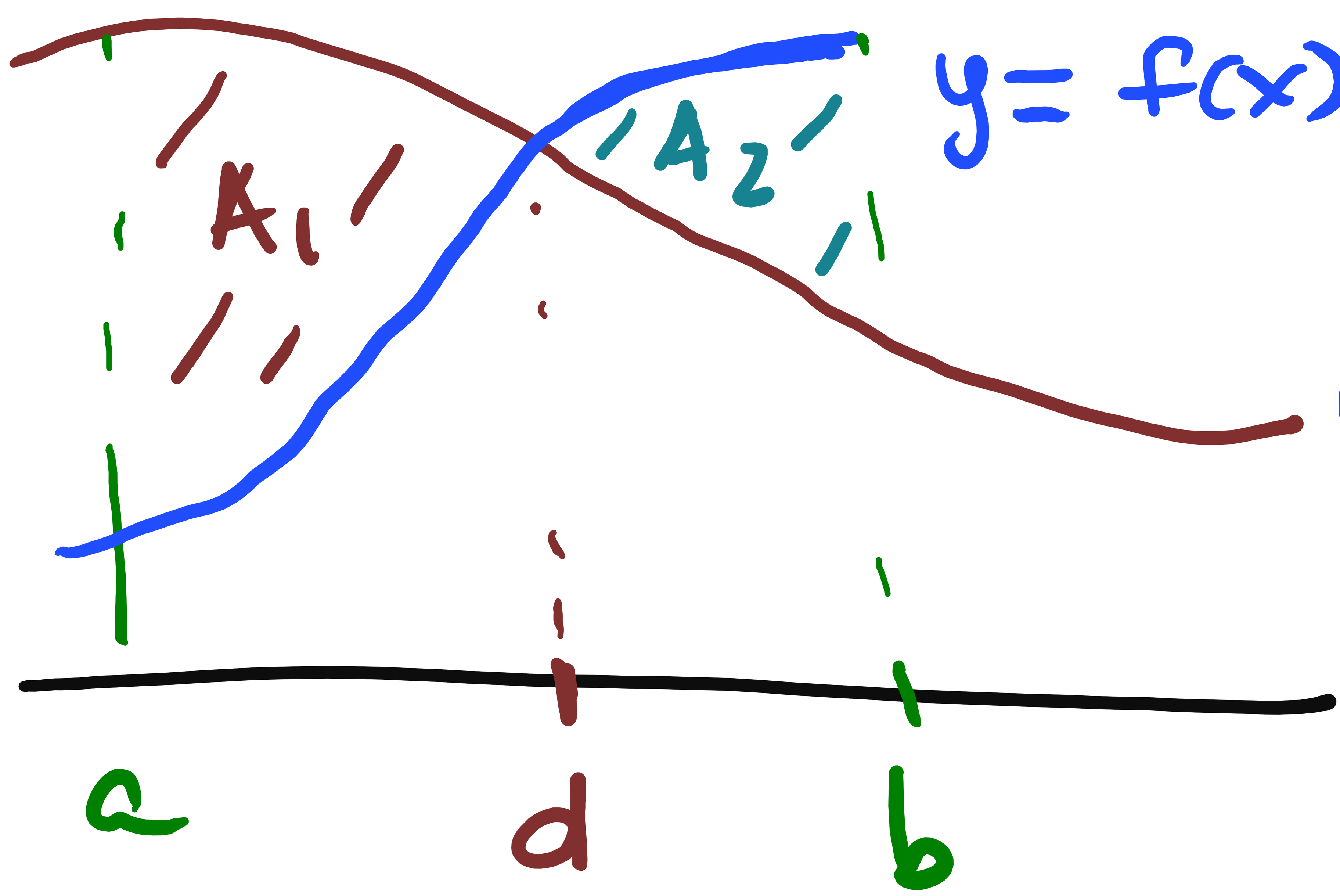




How to find the sum of the areas A_1+A_2, i.e. the areas bounded by the curves $y=f(x)$ and $y=g(x)$, between $x=a$ and $x=b$.

Area A_1 is >0 & Area A_2 is >0.

AREAS are ALWAYS POSITIVE.



Take $C>0$ large enough so translate $f(x)$ & $g(x)$ above x-axis.

This does NOT change the value of A_1+A_2.

$$A_1=\int_a^d \underbrace{(g(x)+\cancel{C})}_{\text{top curve}}-\underbrace{(f(x)+\cancel{C})}_{\text{bottom curve}}\,dx$$

$$A_2=\int_d^b \underbrace{(f(x)+\cancel{C})}_{\text{top curve}}-\underbrace{(g(x)+\cancel{C})}_{\text{bottom curve}}\,dx$$

$$A_2=\int_d^b (f(x)+\cancel{C})-(g(x)+C)\,dx$$
$$=\int_d^b f(x)-g(x)\,dx.$$

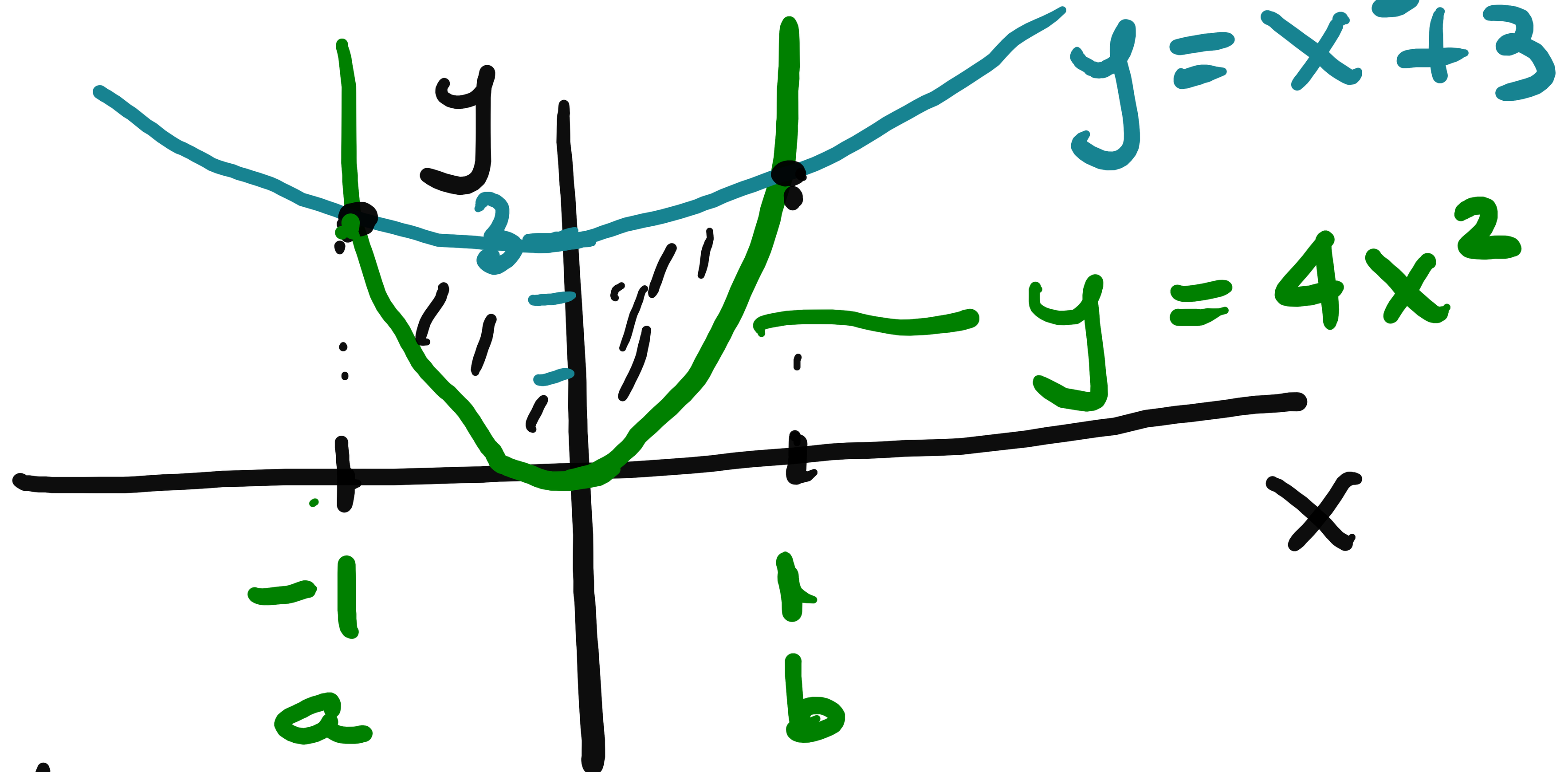
C does not matter

$$A_1 + A_2 = \int_a^b |f(x) - g(x)|\,dx = \int_a^d \underbrace{g(x)}_{\text{top}} - \underbrace{f(x)}_{\text{bottom}}\,dx + \int_d^b \underbrace{f(x)}_{\text{top}} - \underbrace{g(x)}_{\text{bottom}}\,dx$$

Take area under the top curve & subtract the area under the bottom curve.

Example. Find the area of the REGION bounded by the curves $y = x^2+3$ and $y = 4x^2$.

Sol'n Draw a graph if possible.



Find where the curves intersect:

Solve: $4x^2 = x^2+3$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1.$

$$\text{Area} = \int_{-1}^{1} \underbrace{(x^2+3)}_{\text{top}} - \underbrace{(4x^2)}_{\text{bottom}}\,dx$$

$$= \int_{-1}^{1} -3x^2 + 3\,dx$$

$$= 2\int_0^1 -3x^2 + 3\,dx$$

since $-3x^2+3$ is an EVEN function & interval $[-1,1]$ is symmetric.

$$= 2\left(-\frac{\cancel{3}x^3}{\cancel{3}} + 3x \right|_0^1$$

$$= 2\left((-1+3) - (0+0)\right)$$

$$= 2(2) = \underline{4}.$$

<u>The required area is 4.</u>

<u>Example</u> Find the area bounded by the curves $y = \sin(x)$, $y = \cos(2x)$, $x = 0$, and $x = \frac{\pi}{4}$.

<u>Sol'n</u>. Draw the graph:

Find the point of intersection.

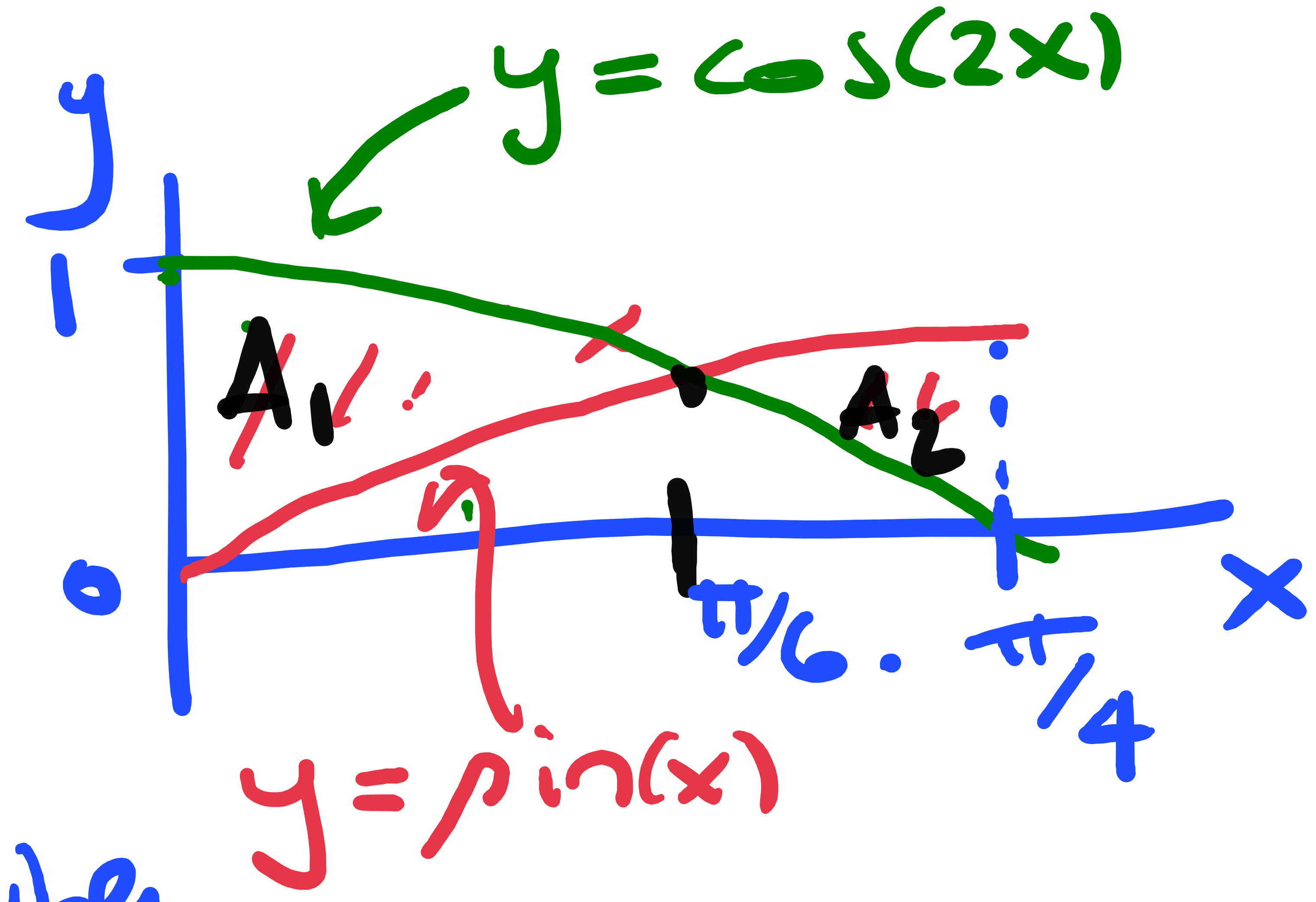
(From the graph there is one.)

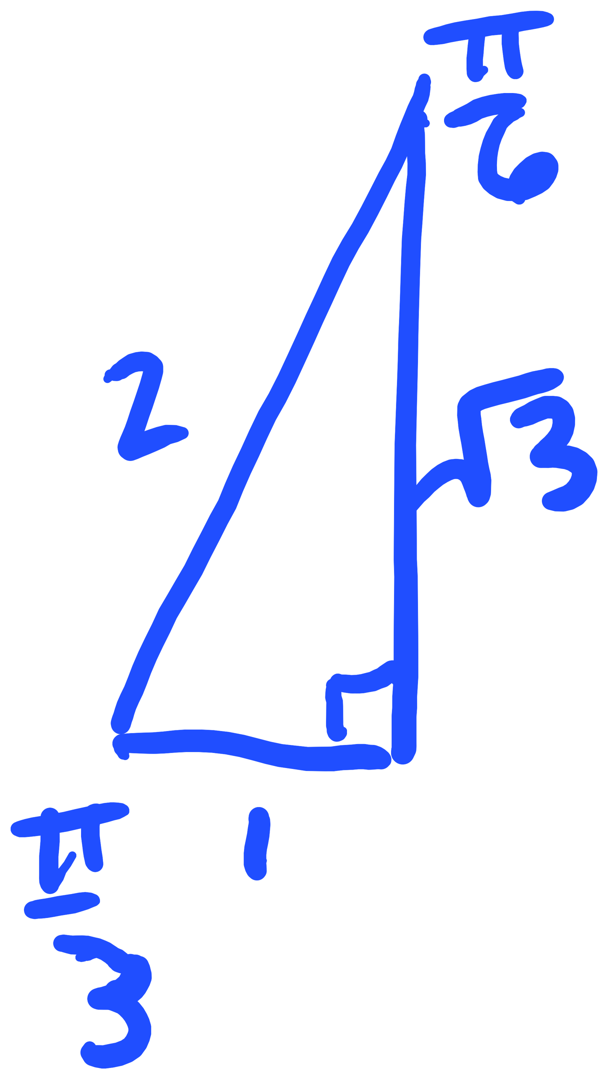
Solve $\sin x = \cos(2x) = 1 - 2\sin^2 x$

$\therefore 2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\therefore \sin(x) = \frac{1}{2}$ or $\underbrace{\sin x = -1}_{\text{but not possible since } 0 \le x \le \pi/4.}$





$\sin(x) = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$ (Need consider only values of $x \in [0, \frac{\pi}{4}]$).

Area $A_1 + A_2$

$$= \int_0^{\pi/4} |\sin(x) - \cos(2x)| \, dx$$

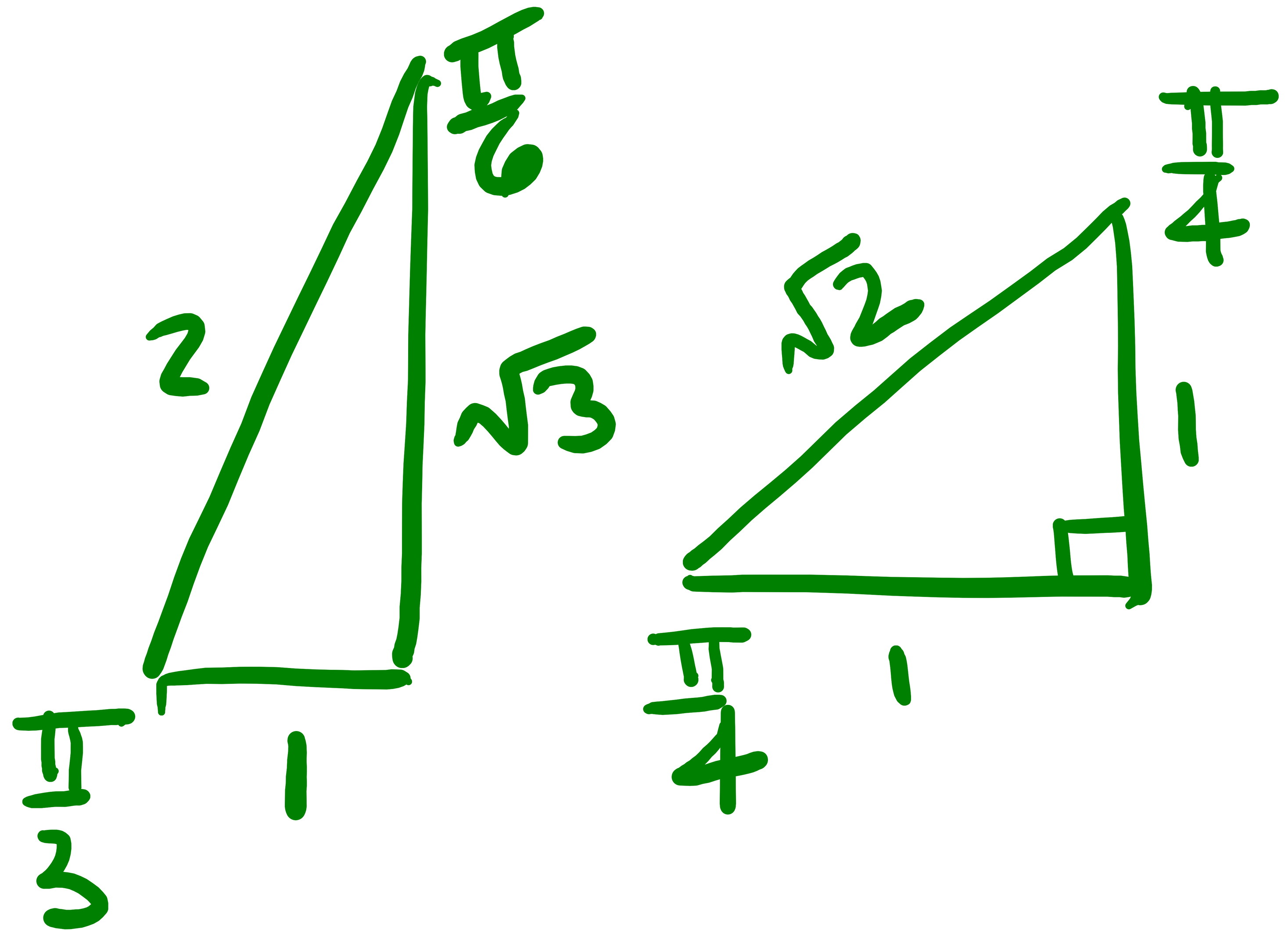
$$= \int_0^{\frac{\pi}{6}} \cos(2x) - \sin(x) \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin(x) - \cos(2x) \, dx$$

$$= \left(\frac{\sin(2x)}{2} - (-\cos(x)) \right) \Big|_0^{\frac{\pi}{6}} + \left(-\cos(x) - \frac{\sin(2x)}{2} \right) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

(Check each term by differentiating to be sure you found the correct anti-deriv.)

$$= \left(\frac{1}{2}\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right) \right) - \left(\frac{1}{2}\sin(0) + \cos(0) \right)$$

$$+ \left(-\cos\left(\frac{\pi}{4}\right) - \frac{\sin\left(\frac{\pi}{2}\right)}{2} \right) - \left(-\cos\left(\frac{\pi}{6}\right) - \frac{\sin\left(\frac{\pi}{3}\right)}{2} \right)$$



$$= \left(\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\right) - \left(0 + 1\right)$$

$$+ \left(-\frac{1}{\sqrt{2}} - \frac{1}{2}(1)\right) - \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= \frac{3}{2}\sqrt{3} - \frac{1}{\sqrt{2}} - \frac{3}{2}.$$

Make sure you do not forget any terms or count any term twice.