M1A03 Calculus I Section C01 Dr. Wolkowicz Lecture 25 November 5, 2019

Information about Test 2 posted on the course Web-site as for test 1

Evaluating Integrals + Sums.

$$\int_a^b f(x)\,dx = \lim_{\|\Delta P\| \to 0} \sum_{i=1}^{n} \Delta x\, f(x_i^*)$$

equal subintervals.

$$\Delta x = \frac{b-a}{n}.$$

$$x_i^* \in [x_{i-1}, x_i]$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f\Big(\underbrace{a + i\,\frac{(b-a)}{n}}\Big)$$

x_i^* is right end pt.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f\Big(\underbrace{a + (i-1)\frac{(b-a)}{n}}\Big)$$

x_i^* is left end point

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f\Big(\underbrace{a + (i-\tfrac{1}{2})\frac{(b-a)}{n}}\Big)$$

x_i^* is mid-point.

Sometimes the sum is difficult to evaluate but the $\int$ is not, and vice versa.

Example: Find $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\frac{2}{n}}_{\Delta x = \frac{2}{n}} \overbrace{\left(4 + \underbrace{\left(1+\frac{2i}{n}\right)}_{x_i^*}^{5}\right)\underbrace{\left(1+\frac{2i}{n}\right)}_{x_i^*}^{25}}^{f(x_i^*)}.$

x_i^* (right end point)

$i=0$ gives a

$\frac{2}{n} = \Delta x = \frac{b-a}{n}$.

$x_i^* = \left(1+\frac{2i}{n}\right)$ $\quad i=0 \Rightarrow a=1.$

$\frac{b-a}{n} = \frac{2}{n} \Rightarrow b=3$

$\left(\int x^n\,dx = \frac{x^{n+1}}{n+1} + C\right).$

$$= \int_1^3 (4+x^5)(x)^{25}\,dx$$

$$= \int_1^3 4x^{25} + x^{30}\,dx$$

$$= \left(4\,\frac{x^{26}}{26} + \frac{x^{31}}{31}\right)\Bigg|_1^3$$

$$= \left(4\,\frac{(3)^{26}}{26} + \frac{3^{31}}{31}\right) - \left(\frac{4}{26} + \frac{1}{31}\right)$$

Find the mistake.

$$\int_{-1}^{2} \frac{3}{x^4}\,dx = \int_{-1}^{2} 3x^{-4}\,dx$$

$$= 3\,\frac{x^{-3}}{-3}\Bigg|_{-1}^{2} = -x^{-3}\Big|_{-1}^{2}$$

$$= -\left(x^{-3}\Big|_{-1}^{2}\right.$$

$$= -\left(2^{-3} - (-1)^{-3}\right)$$

$$= -\left(\frac{1}{8}+1\right) = -\frac{9}{8}.$$

impossible, since $3/x^4 > 0$ so $\int$ should be positive.

WRONG because $f(x) = \frac{3}{x^4}$ is NOT continuous at $x=0$ and $0 \in [-2,1]$..

To use the FTC P2, $f(x)$ must be contin. in the entire closed interval.

§5.5 The Substitution Rule.

Finding anti-derivatives is not always easy! We need methods of integration.

$$\underbrace{\int_a^b f(x)\,dx}_{\text{definite integral}} = \underbrace{\int f(x)\,dx}_{\substack{\text{indefinite integral} \\ \text{an anti-derivative.}}}\Bigg|_a^b \quad \text{by FTC P2.}$$

<u>Substitution Rule</u>

- based on the chain rule.

$$\frac{d}{dx} F(g(x)) = F'(g(x))\,g'(x)$$

Ⓐ $$\int F'(g(x))\,g'(x)\,dx = F(g(x))$$

check by differentiating

Make a "substitution" or "a "change of variables."

$u = g(x) \implies \frac{du}{dx} = g'(x)$

$du = g'(x)\,dx$

↑ differential.

$$\int F'(g(x))\,g'(x)\,dx$$

$$= \int F'(u)\,du$$

or equivalently

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du.$$

B.

use this if is easier to integrate than the original.

Example!: $\int \sqrt{\tan x}\,\sec^2 x\,dx$

Let $u = \tan x$

$du = \sec^2 x \, dx$

$$\int \underbrace{\sqrt{\tan x}}_{\sqrt{u}} \, \underbrace{\sec^2 x \, dx}_{du} = \int u^{1/2} \, du$$

$$= \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{3} (\tan x)^{3/2} + C.$$

Example: $\int x e^{x^2} dx$

Let $u = x^2$

$du = 2x \, dx \qquad \frac{1}{2} du = x \, dx$

$$\int x e^{x^2} dx = \int e^{u} \left(\tfrac{1}{2}\right) du = \frac{1}{2} e^{u} + C = \frac{1}{2} e^{x^2} + C.$$

Example. $$\int \frac{2x^4}{3+x^5}\,dx$$

Let

$$u = 3+x^5$$
$$du = 5x^4\,dx \qquad \frac{2}{5}\,du = 2x^4\,dx$$

$$\int \frac{2x^4}{3+x^5}\,dx = \frac{2}{5}\int \frac{1}{u}\,du$$
$$= \frac{2}{5}\ln|u| + C$$
$$= \frac{2}{5}\ln(3+x^5) + C.$$

$$\int k u\,du = k\int u\,du$$

alternatively

$$u = x^5$$
$$du = 5x^4\,dx$$

$$\int \frac{2x^4}{3+x^5}\,dx = \frac{2}{3}\int \frac{1}{3+u}\,du$$
$$= \frac{2}{5}\ln|3+u| + C$$
$$= 2/5\ln|3+x^5| + C.$$

What about definite integrals.

$$\int_{x=a}^{x=b} \underbrace{f(g(x))}_{f(u)} \underbrace{g'(x)\,dx}_{du} = \int_{u=g(a)}^{u=g(b)} f(u)\,du$$

Must also change limits of integration.

$$u = g(x)$$

$$du = g'(x)\,dx$$

$$\therefore \int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example $\displaystyle\int_{x=e}^{e^2} \frac{1}{x\ln x}\,dx$

Let $u = \ln x$

$du = \frac{1}{x}dx$

$u = \ln x$

$x = e$

$\Rightarrow$

$u = \ln e$

$= 1.$

$$= \int_{u=\ln(e)}^{u=\ln(e^2)} \frac{1}{u}\,du = \int_1^2 \frac{1}{u}\,du = \ln|u|\Big|_1^2$$

$$= \ln 2 - \ln 1 = \underline{\ln 2}.$$

OR $= \ln|u| \Big|_{u=1}^{u=2}$

change back to x:

$= \ln(\ln(x)) \Big|_{x=e}^{x=e^2}$

$= \ln(\ln e^2) - \ln(\ln(e))$

$= \ln(2\ln e) - \ln(1)$

$= \underline{\ln(2)}$.

BEWARE: Pay attention to the limits of integration.

Evaluate using the correct variable.