Information about Test 2 posted on the course Web-site as for test 1

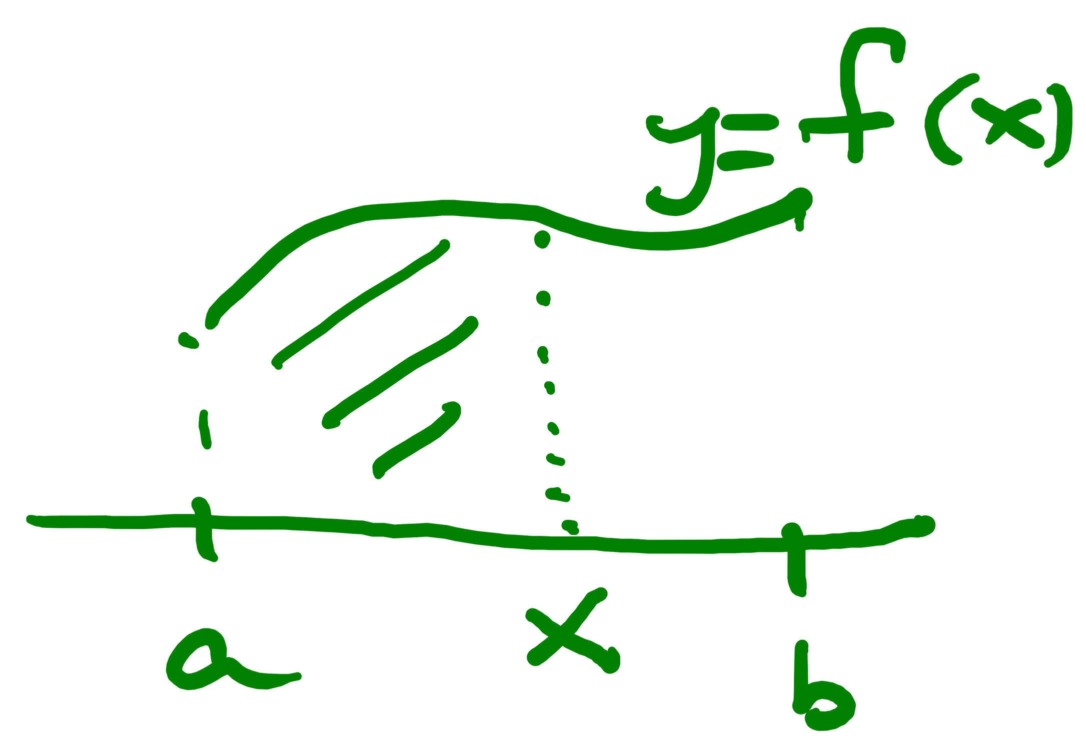
§5.3 The FUNDAMENTAL THEORM of CALCULUS (FTC).

Part 1: If f is continuous on $[a,b]$ and

$$g(x) = \int_a^x f(t)\,dt,$$

then $g'(x) = f(x)$.

i.e. $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$



$g(x)$ is the area under $f(x)$ from a to x

Example:

$$\frac{d}{dx}\int_a^x \sin(t^2)\,dt = \sin(x^2)$$

Proof of Part I of FTC.

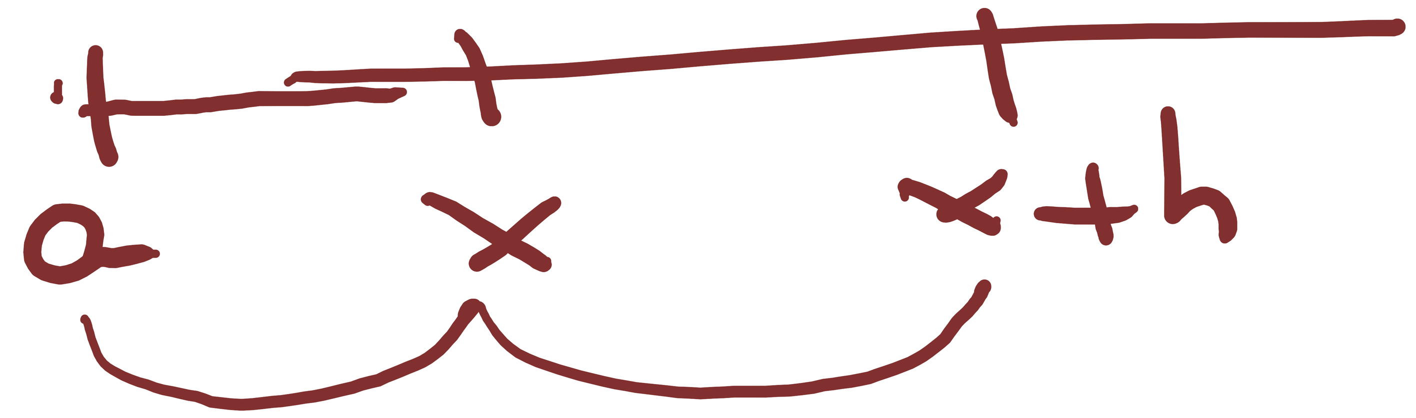
Let $g(x) = \int_a^x f(t)\,dt$

To find $g'(x)$

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \left(\cancel{\int_a^x f(t)\,dt} + \int_x^{x+h} f(t)\,dt\right) - \cancel{\int_a^x f(t)\,dt}$$

$$g(x+h)-g(x) = \int_x^{x+h} f(t)\,dt$$



$$\frac{d}{dx} g(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$$

Case 1: Assume $h>0$. (Case for $h<0$ has a similar proof.)

By the Extreme Value Thm, since $f(x)$ is continuous on $[a, b]$, there exists numbers u, v in $[a, b]$ such that

$$f(u) = m \quad \text{and} \quad f(v) = M$$

where m is the absolute min value of f on $[a,b]$
M " max " f on $[a,b]$.

$$f(u) = m \le f(x) \le M = f(v) \quad \text{for all } x \in [a,b].$$

Comparing.

$$f(u)h = mh \le \int_x^{x+h} f(t)\,dt \le \underset{\substack{\parallel \\ f(v)h}}{Mh}.$$

$m \le f(x) \le M.$

$m(b-a) \le \int_a^b f(t)\,dt \le M(b-a).$

Divide by $h > 0$

$$f(u) \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le f(v)$$

$$f(u) \le \frac{g(x+h) - g(x)}{h} \le f(v) \quad (*)$$

Since $u, v \in [x, x+h]$ as $h \to 0$ $f(u) \to f(x)$
$f(v) \to f(x)$

Let $h \to 0$ in $(*)$

$$\lim_{h\to 0} f(u) \leq \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \leq \lim_{h\to 0} f(v)$$

$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$f(x) \qquad\qquad\qquad g'(x) \qquad\qquad\qquad f(x)$$

By the Squeeze Th'm.

$$f(x) = g'(x).$$

i.e. $g'(x) = f(x)$

Example $\displaystyle \frac{d}{dx}\int_a^x \arctan(\cos^3(t)) + e^{t^2}\,dt$

$\displaystyle = \arctan(\cos^3(x)) + e^{x^2}$ by FTC P1.

Example $\displaystyle \frac{d}{dx}\int_x^a f(t)\,dt = \frac{d}{dx}\left(-\int_a^x f(t)\,dt\right)$

$= -f(x).$

Combining FTC P1 with the chain rule.

If $g(x) = \int_a^x f(t)\,dt$. $\therefore g'(x) = f(x)$, $g'(h(x)) = f(h(x))$

If $g(h(x)) = \int_a^{h(x)} f(t)\,dt$

$$\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = \frac{d}{dx} g(h(x)) = g'(h(x))\,h'(x) = f(h(x))\,h'(x)$$

$$\therefore \boxed{\frac{d}{dx}\int_a^{h(x)} f(t)\,dt = f(h(x))\,h'(x).}$$ chain rule & FTC P1.

Corollary

$$\frac{d}{dx}\int_{h(x)}^{b} f(t)\,dt = -\frac{d}{dx}\int_b^{h(x)} f(t)\,dt = -f(h(x))\,h'(x).$$

Example

$$\frac{d}{dx}\int_{x^3}^{x^2} \frac{\cos(\theta)}{\theta}\,d\theta = \frac{d}{dx}\left[\int_{x^3}^{a} \frac{\cos(\theta)}{\theta}\,d\theta + \int_{a}^{x^2} \frac{\cos\theta}{\theta}\,d\theta\right]$$

$$= -\frac{d}{dx}\int_{a}^{x^3} \frac{\cos\theta}{\theta}\,d\theta + \int_{a}^{x^2} \frac{\cos\theta}{\theta}\,d\theta$$

$$= -\frac{\cos(x^3)}{x^3}\,\frac{d(x^3)}{dx} + \int_{a}^{x^2} \frac{\cos(x^2)}{x^2}\,\frac{dx^2}{dx}$$

$$= -\frac{\cos(x^3)}{x^3}(3x^2) + \frac{\cos(x^2)}{x^2}(2x)$$

$$= -\frac{3\cos(x^3)}{x} + \frac{2\cos(x^2)}{x}.$$

Example. $\frac{d}{dx}\int_x^{\ln x} \frac{\tan\theta}{\theta}\,d\theta$

$$= \frac{d}{dx}\left(\int_x^a \frac{\tan\theta}{\theta}\,d\theta + \int_a^{\ln x}\frac{\tan\theta}{\theta}\,d\theta\right)$$

$$= -\frac{d}{dx}\int_a^x \frac{\tan\theta}{\theta}\,d\theta + \frac{d}{dx}\int_a^{\ln x}\frac{\tan\theta}{\theta}\,d\theta$$

$$= -\frac{\tan x}{x} + \frac{\tan(\ln x)}{\ln(x)}\frac{d\ln x}{dx}$$

$$= -\frac{\tan x}{x} + \frac{\tan(\ln x)}{x\ln(x)}$$

Thm FUNDAMENTAL THM of Calculus Part II.

If f is continuous on $[a,b]$,

then $\int_a^b f(x)\,dx = F(b) - F(a)$

or $\int_a^b f'(x)\,dx = f(b) - f(a)$

where F is any antideriv of f.

i.e. $F'(x) = f(x)$.

Proof: Let $g(x) = \int_a^x f(t)\,dt$ (*)

Then $g'(x) = f(x)$ (FTC Part 1)

i.e. g is an antiderivative of f.

Let F be any antiderivative of f.

Then $F(x) = g(x) + C$ for some constant C.

$F(a) = g(a) + C = \int_a^a f(t)\,dt + C = 0 + C = C.$

$$F(b) - F(a) = (g(b) + c) - F(a)$$
$$= g(b) + c - c.$$
$$= g(b)$$
$$= \int_a^b f(t)\,dt.$$

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

This is why we use $\underbrace{\int f(x)\,dx}_{\text{indefinite integral.}}$ for the antiderivative of f.

Recall. $\int f(x)\,dx = F(x) + C$, C arb. constant.

& $F'(x) = f(x).$

Notation. $F(b) - F(a) = F\Big|_a^b$ or $F(x)\Big|_{x=a}^{x=b}$

Example (use FTC part 2)

$$\int_1^3 x^3 + 4x + e^{2x}\,dx.$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1.$$

$$\int e^{2x}\,dx = \frac{e^{2x}}{2} + C$$

$$= \left(\frac{x^4}{4} + 4\frac{x^2}{2} + \frac{e^{2x}}{2}\right\Bigg|_{x=1}^{3}$$

$$= \left(\frac{3^4}{4} + \frac{4(3)^2}{2} + \frac{e^6}{2}\right) - \left(\frac{1}{4} + \frac{4}{2} + \frac{e^2}{2}\right)$$

$$= \frac{81}{4}. + \frac{36}{2} + \frac{e^6}{2} - \frac{1}{4} - 2 - \frac{e^2}{2}$$