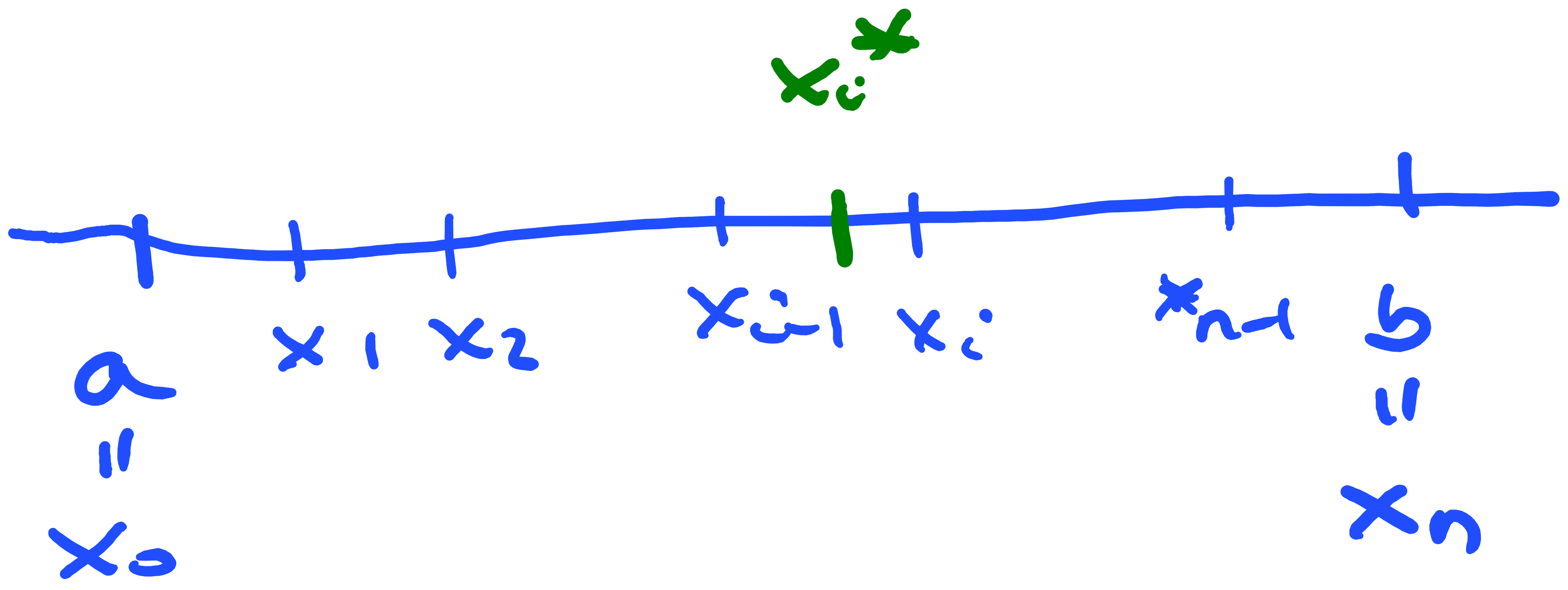
§ 5.2 cont'd Definite Integral.

RECALL: $\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$

(if the limit exists).

Partition P. of $[a,b]$ $\quad a \le x_0 < x_1 < \cdots < x_n = b$



$x_i^* \in [x_{i-1}, x_i]$

$\Delta x_i = x_i - x_{i-1}$

$\|P\| = \max\{\Delta x_i\}$.

x_i^* left end point, x_{i-1}

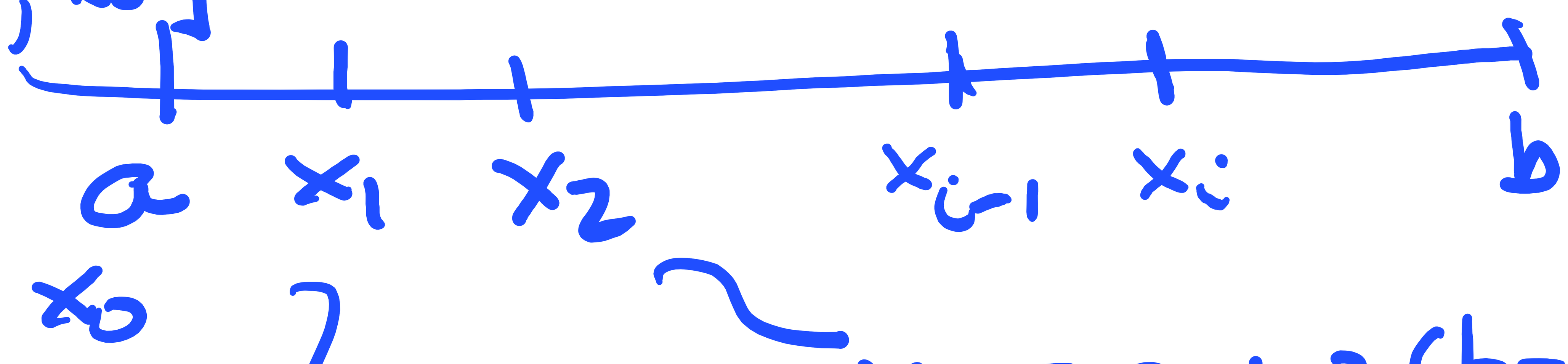
x_i^* right end point, x_i

midpoint $\dfrac{x_i + x_{i-1}}{2}$.

or any other convenient pt. in the interval.

In the case of subintervals of equal length they all have length: $\Delta x_i = \dfrac{b-a}{n}$ for each $i = 1, 2, \ldots, n$.

RIGHT end point of each $[x_{i-1}, x_i]$



$$x_i^* = a + i\,\frac{(b-a)}{n}$$

$$i = 1, 2, \ldots, n$$

$$x_1 = a + \frac{b-a}{n}$$

$$x_2 = a + 2\,\frac{(b-a)}{n}$$

LEFT end point of each $[x_{i-1}, x_i]$

$$x_i^* = a + (i-1)\,\frac{(b-a)}{n}$$

$$i = 1, 2, \ldots, n.$$

Example: Write $\displaystyle \lim_{\|P\| \to 0} \sum_{i=1}^{n} \underbrace{x_i^* \tan(x_i^*)}_{f(x_i^*)} \Delta x_i$

as an integral on $[0, 2\pi]$.

It is equal to $\displaystyle \int_0^{2\pi} \underbrace{x \tan(x)}_{f(x)}\, dx$

Note: If $f(x) \geq 0$ on $[a, b]$

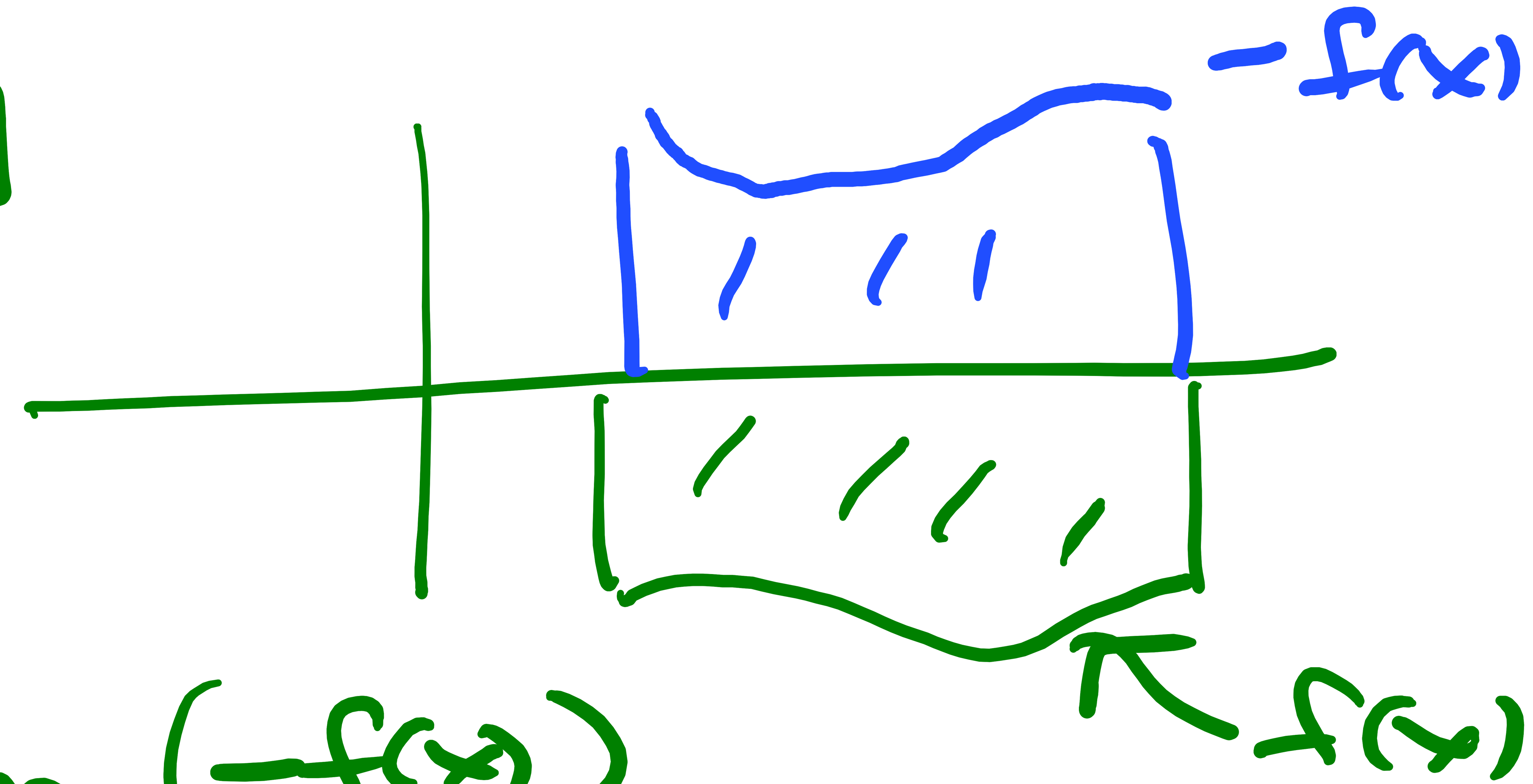
then $\displaystyle \int_a^b f(x)\,dx \geq 0$ and is equal to the area under the curve

But it can represent other things.

$y = f(x)$ from a to b.

Properties. 1. If $f(x) \le 0$ on $[a, b]$

$$\int_a^b f(x)\,dx = -\int_a^b \underbrace{-f(x)}_{\ge 0}\,dx$$



$= -$ Area under $(-f(x))$

≤ 0. i.e. it is $-$(AREA) above the curve ≤ 0.

2. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

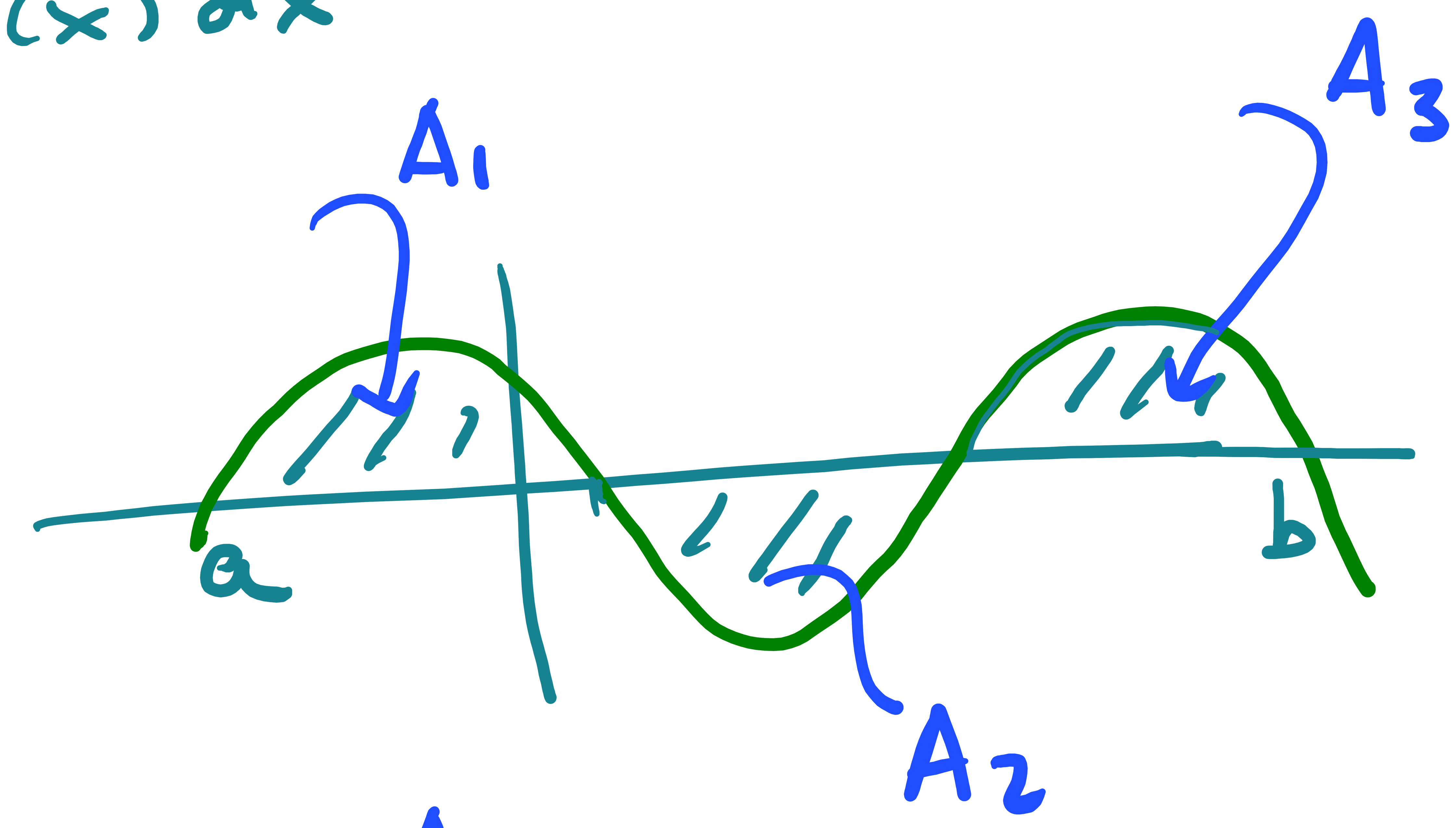
(interchanging limits of integration changes the sign.

3. If $a < b$

$$\int_a^b f(x)\,dx$$

$= \text{area } A_1 + \text{area } A_3$

$- \text{area } A_2$



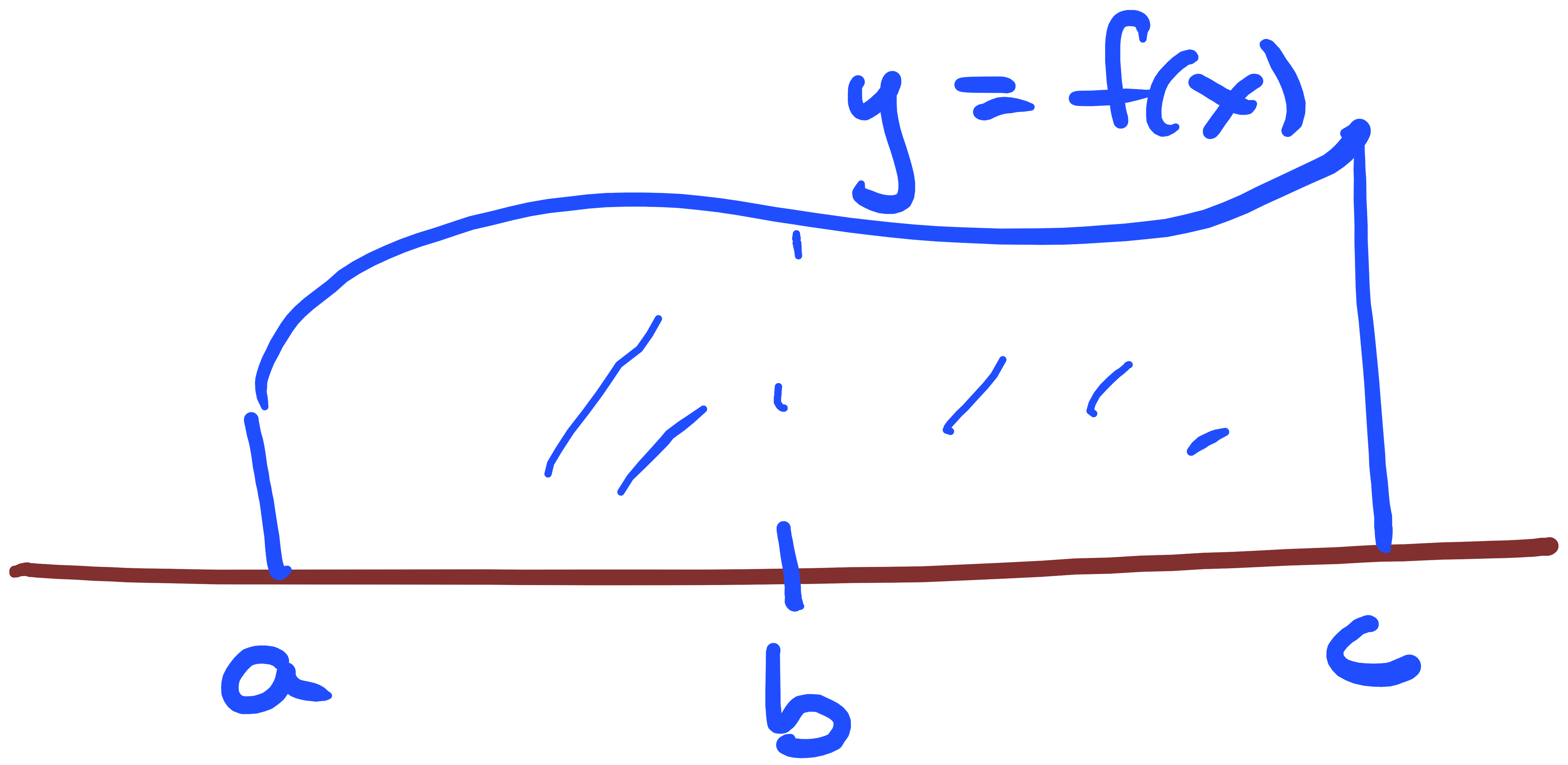
$A_i > 0$

$i = 1, 2, 3$.

4. $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

5. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

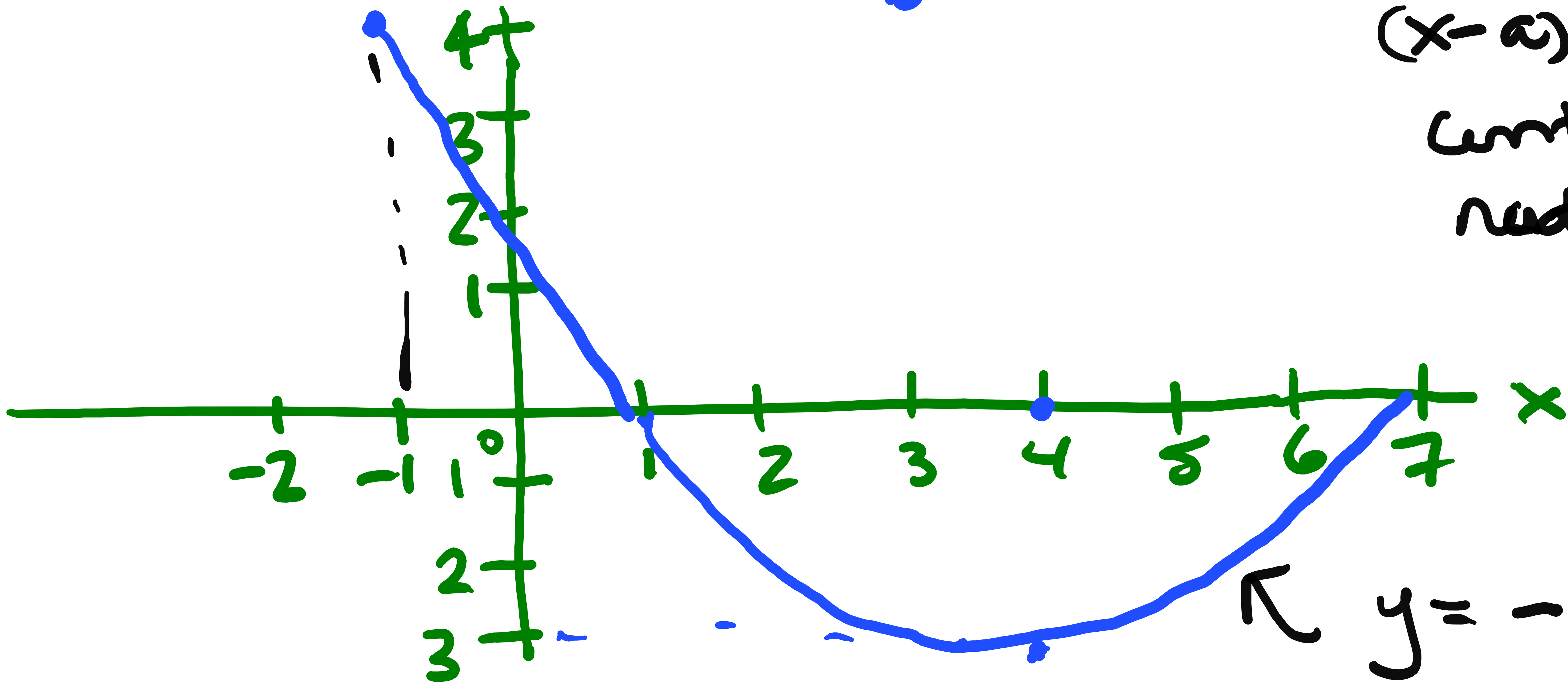
6. $\int_a^a f(x)\,dx = 0$



7. $\int_a^c f(x)\,dx$

$$= \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Example



Circle

$(x-a)^2 + (y-b)^2 = r^2$

centre (a,b)

radius r.

Area of the circle : πr^2

$y = -\sqrt{9-(x-4)^2}$

Find $\int_{-1}^{7} f(x)\,dx$ where $f(x) = \begin{cases} -2x+2 & -1 \le x \le 1 \\ -\sqrt{9-(x-4)^2} & 1 \le x \le 7. \end{cases}$

$= \text{Area of the } \Delta$

$\quad - \text{ area of the semicircle.}$

CIRCLE with centre $(x,y) = (4,0)$ and radius $r=3$ has equation:

$$(x-4)^2 + y^2 = 9$$

$$y^2 = 9-(x-4)^2$$

$$y = \pm\sqrt{9-(x-4)^2}$$

The lower semicircle has equation

$$y = -\sqrt{9-(x-4)^2}$$

$= \frac{1}{2}\,\text{base height} - \frac{1}{2}\pi r^2$

$= \frac{1}{2}(2)(4) - \frac{1}{2}\pi 3^2$

$= 4 - \frac{9}{2}\pi \quad \left(= \int_{-1}^{1} f(x)\,dx + \int_{1}^{7} f(x)\,dx\right)$

Comparison Properties.

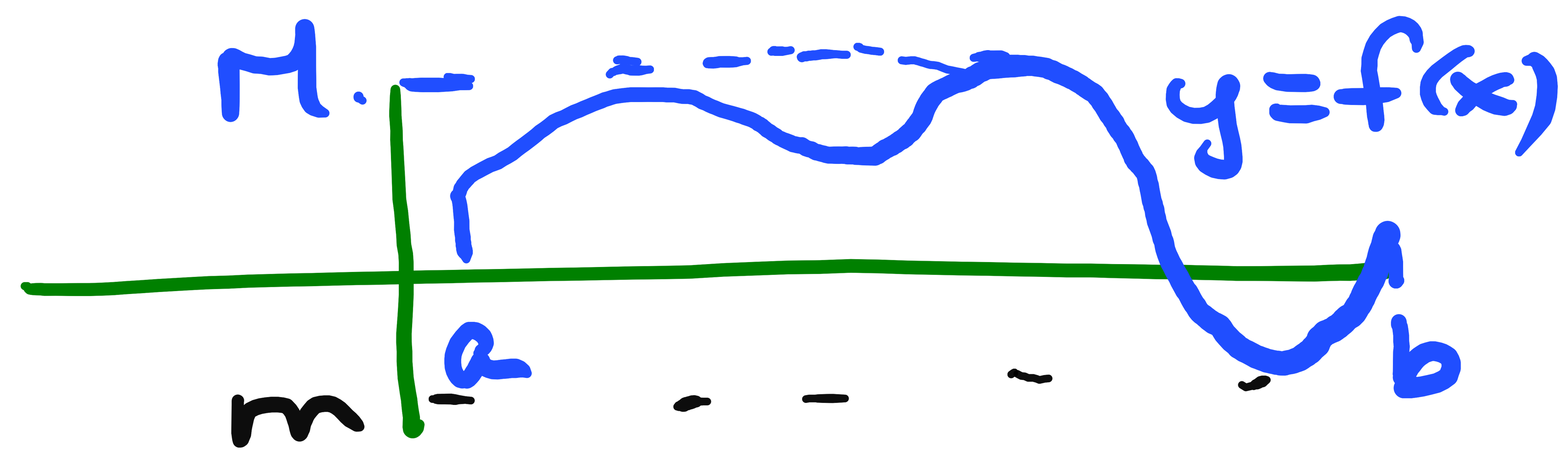
$a \le x \le b.$

(a) If $f(x) \geq g(x)$ $x \in [a,b]$.

then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

(b). If $m \le f(x) \le M$, $a \le x \le b$.

$\Rightarrow \underbrace{m(b-a)}_{\text{area of rectangle.}} \le \int_a^b f(x)\,dx \le \underbrace{M(b-a)}_{\text{area of rectangle.}}$



Example. Given. $\int_1^5 g(x)\,dx = 4$; $\int_1^3 f(x)\,dx = 2$

$\int_5^3 f(x)\,dx = 13$.

Evaluate: $\int_1^5 2g(x) - f(x)\,dx$

$$= \int_1^5 2g(x)\,dx - \int_1^5 f(x)\,dx$$

$\left\{ \int_a^b u(x) - v(x)\,dx = \int_a^b u(x)\,dx - \int_a^b v(x)\,dx \right.$

$$= 2\int_1^5 g(x)\,dx - \left\{ \int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx \right\}$$

$\left\{ \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \right.$

$$= 2(4) - \left\{ 2 - \int_5^3 f(x)\,dx \right\}$$

$\left\{ \int_a^b u(x)\,dx = -\int_b^a u(x)\,dx \right.$

$$= 8 - \{2 - 13\}$$

$$= 8 - 2 + 13$$

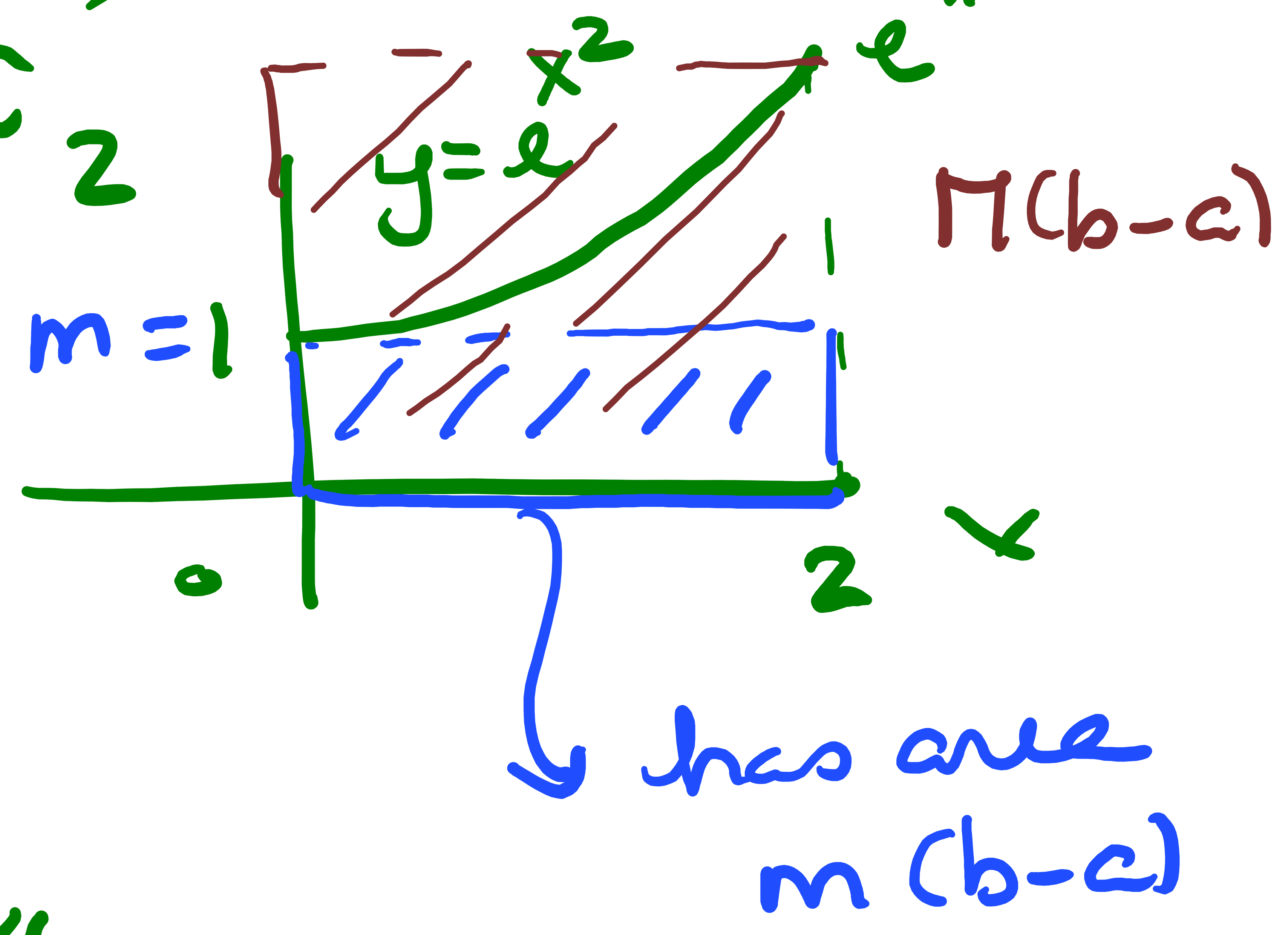
$$= 19.$$

a c b

Estimate $\int_0^2 e^{x^2}\,dx$.

$m \le f(x) \le M.$ for $\underset{a}{0} \le x \le \underset{b.}{2}$

$\underset{\uparrow\, 2}{m(b-a)} \le \int_a^b f(x)\,dx \le \underset{\uparrow\, 2}{M(b-a)}$



e^{x^2} is an increasing for

$m = e^0 = 1$

$M = e^{2^2} = e^4$

$$\underbrace{\underset{2m}{2}}_{\text{underestimate}} \le \int_0^2 e^{x^2} \le \underbrace{\underset{2M}{2e^4}}_{\text{overestimate}}.$$