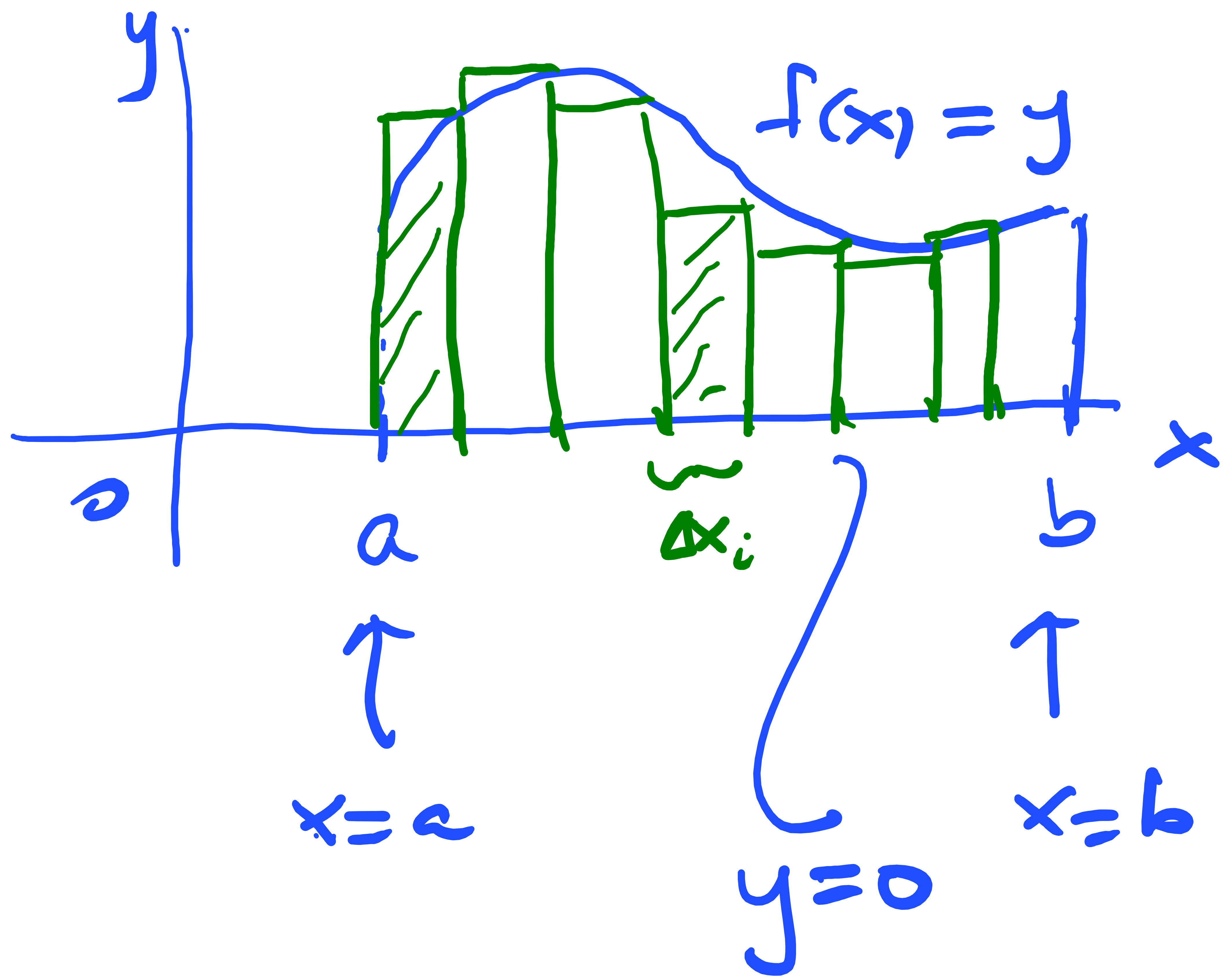
§5.1 Areas

Area Problem

If $f(x) \geq 0$, $a \leq x \leq b$
i.e. $x \in [a,b]$



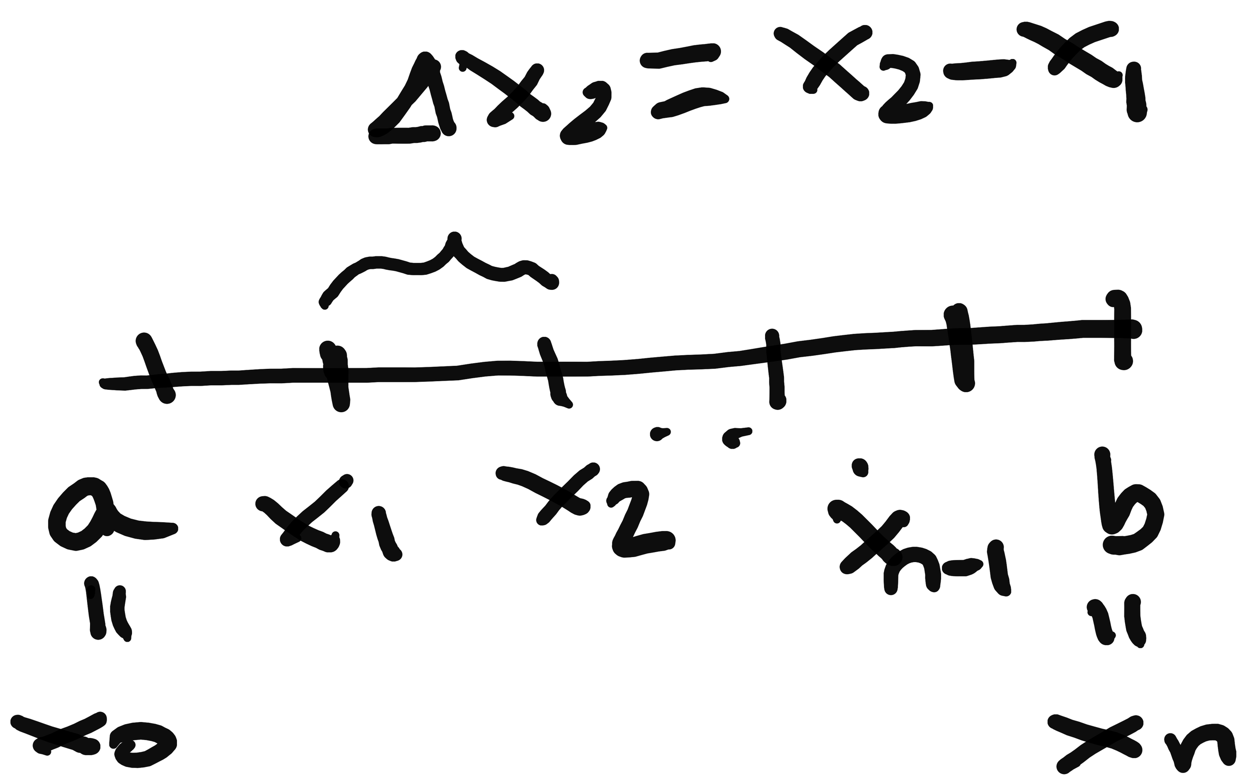
Find the area A of the region bounded by

$$y = f(x)$$
$$y = 0$$
$$x = a \ \& \ x = b$$

i.e. the area under $f(x)$, above the x axis between $x=a$ and $x=b$.

Let P be a partition of $[a,b]$ into n subintervals $[x_{i-1}, x_i]$

$$a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_n.$$

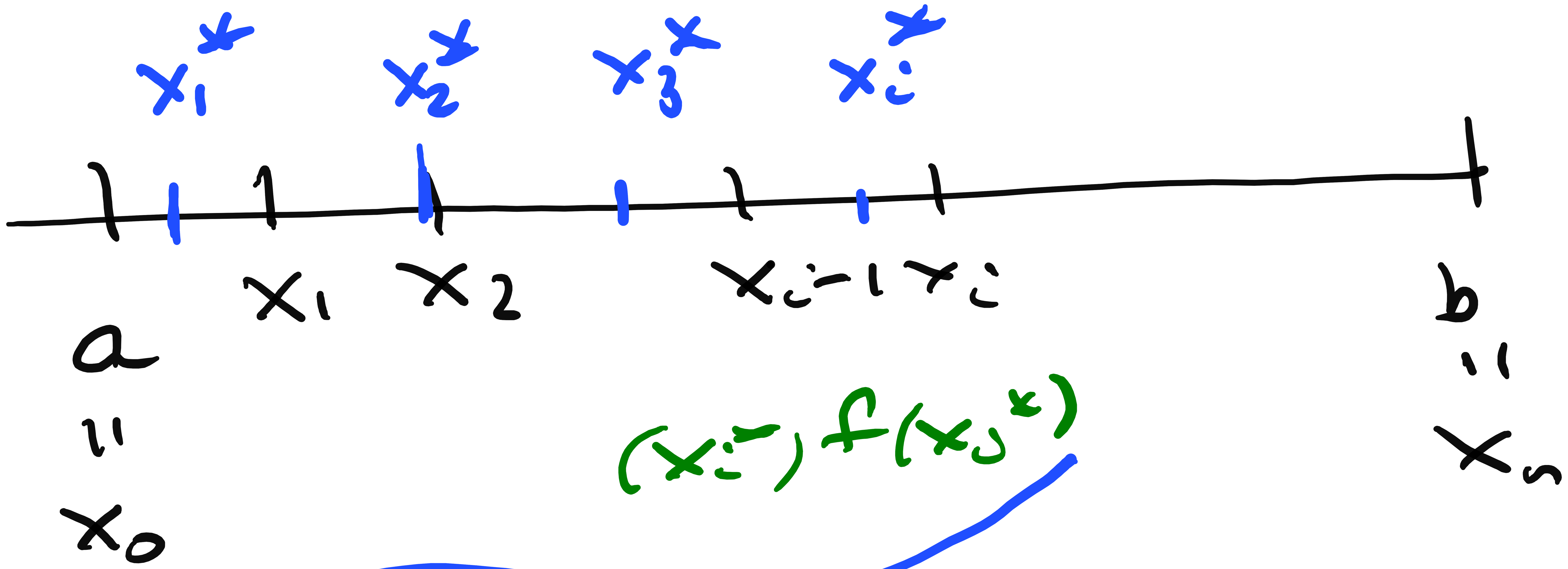


Let $\Delta x_i = x_i - x_{i-1}$ (length of the i^{th} subinterval.

Let $\|P\| = \max\limits_{i=1,2,\dots,n}\{\Delta x_i\}$ (The norm of P).

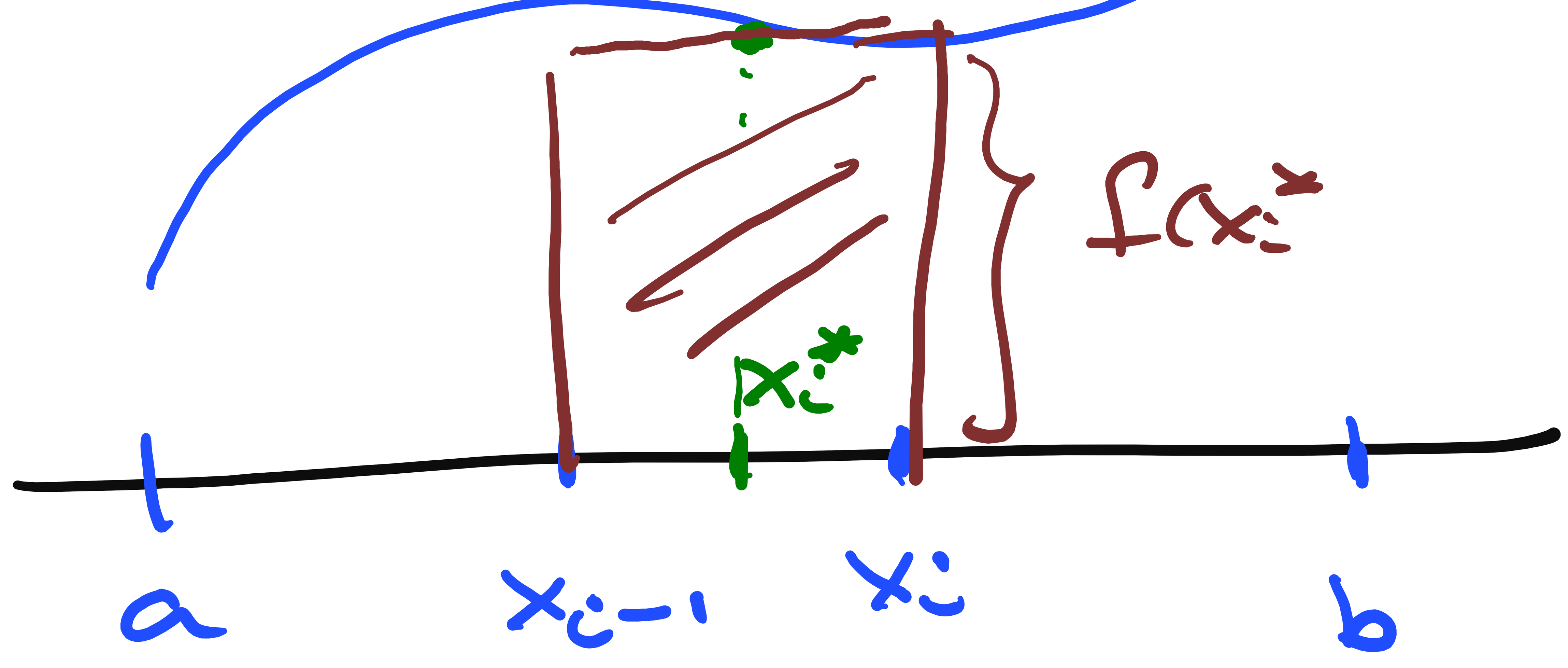
Choose any number $x_i^* \in [x_{i-1}, x_i]$

$(x_{i-1} \le x_i^* \le x_i)$



Area of the i^{th} strip.

$$\approx f(x_i^*)\Delta x_i$$



$\Delta x_i = x_i - x_{i-1}$

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \dots, \Delta x_n\}$$

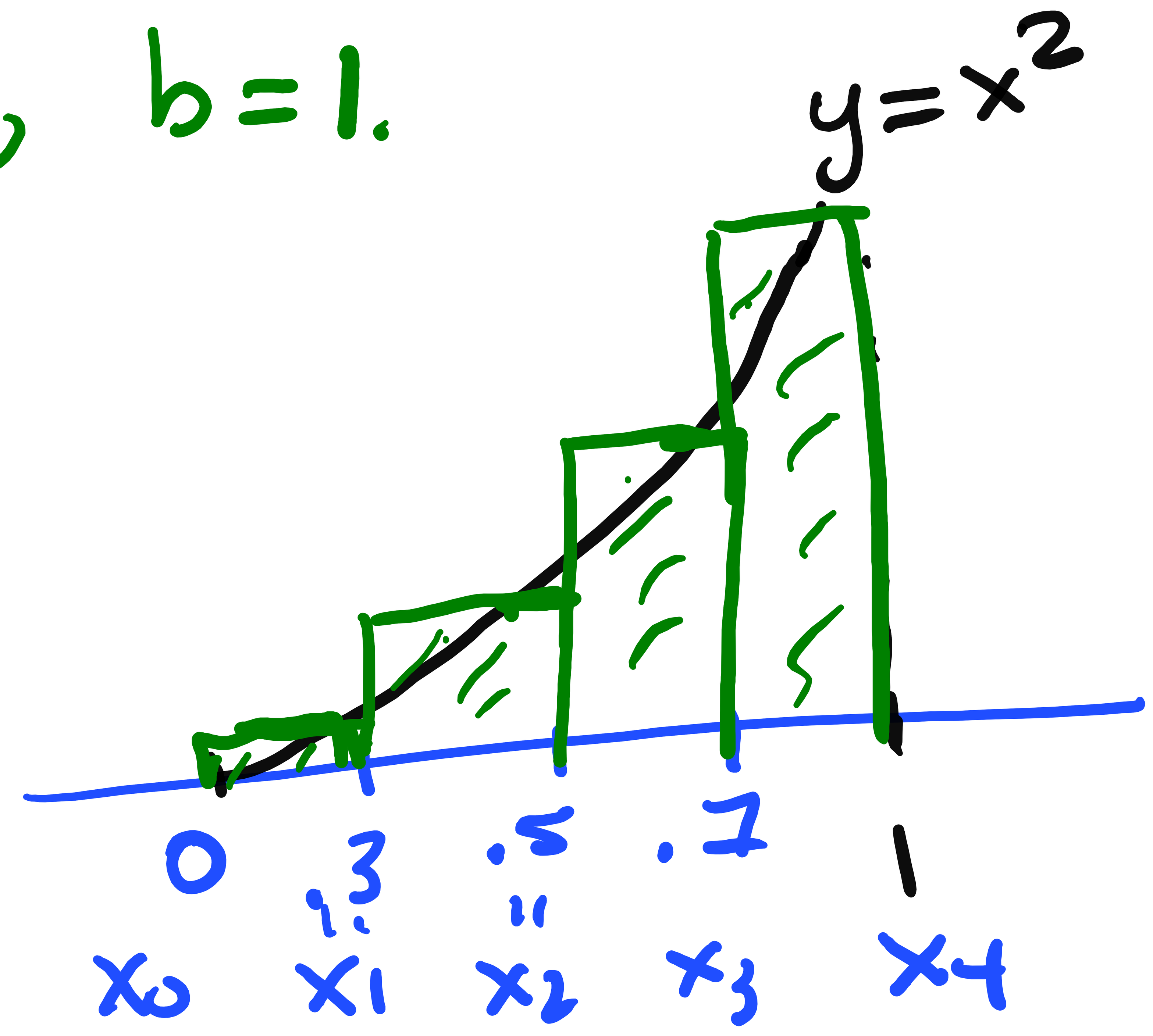
"total area" under $f(x)$ from a to b is approximately

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

Example: $f(x) = x^2$, $a = 0$, $b = 1$.

Partition

$x_0 = 0$, $x_1 = 0.3$, $x_2 = 0.5$, $x_3 = 0.7$,
$x_4 = 1$



$n = 4$.

$\Delta x_1 = .3 - 0 = .3$

$\Delta x_2 = .5 - .3 = .2$

$\Delta x_3 = .7 - .5 = .2$

$\Delta x_4 = 1 - .7 = .3$

$\|P\| = \max\{\Delta x_i\} = .3$.

Choose $x_1^* = 0.2$, $x_2^* = 0.4$, $x_3^* = 0.6$, $x_4^* = 1$.

The sum of the area of the approximating rectangles

$$\sum_{i=1}^{4} f(x_i^*)\,\Delta x_i$$

$$= f(0.2)(.3) + f(0.4)(.2) + f(0.6)(.2) + f(1)(.3)$$

$f(x) = x^2$

$$= (.04)(.3) + (.16)(.2) + (0.36)(.2) + 1(.3)$$

$$= .012 + .032 + .072 + .3$$

$$= 0.416.$$

Letting $\|P\| \to 0$, i.e. letting $n \to \infty$
i.e. increasing the number of points in P,

the approximation gets better and better and in the limit as $\|P\| \to 0$ one obtains the actual area (not just an approximation).

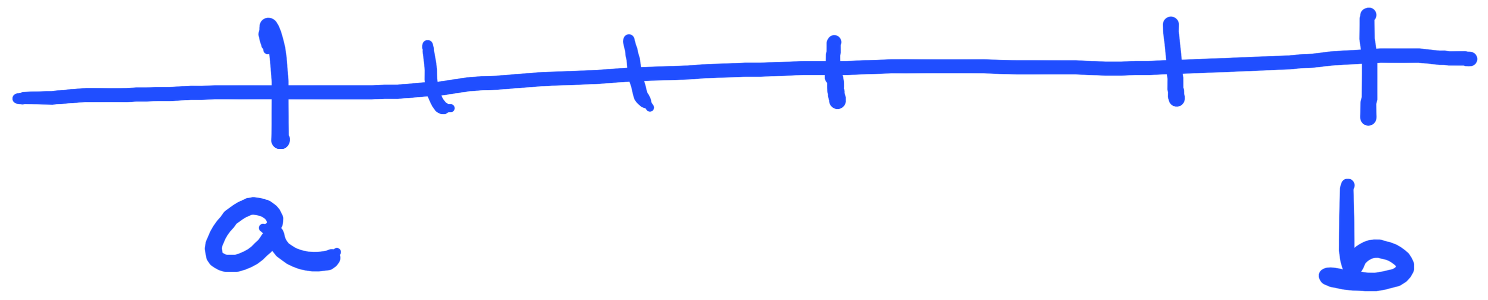
Def'n If $f(x) \geq 0$, the area under $y = f(x)$ from a to b above the x-axis is:

$$A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

(i.e. $n \to \infty$).

Example: Find the AREA under $y = x^2$ from 0 to 1 using equal length subintervals and take $x_i^* =$ the right end point of the i^{th} interval $(i = 1, 2, \ldots, n)$.

Interval of equal length on $[a,b]$ each has length



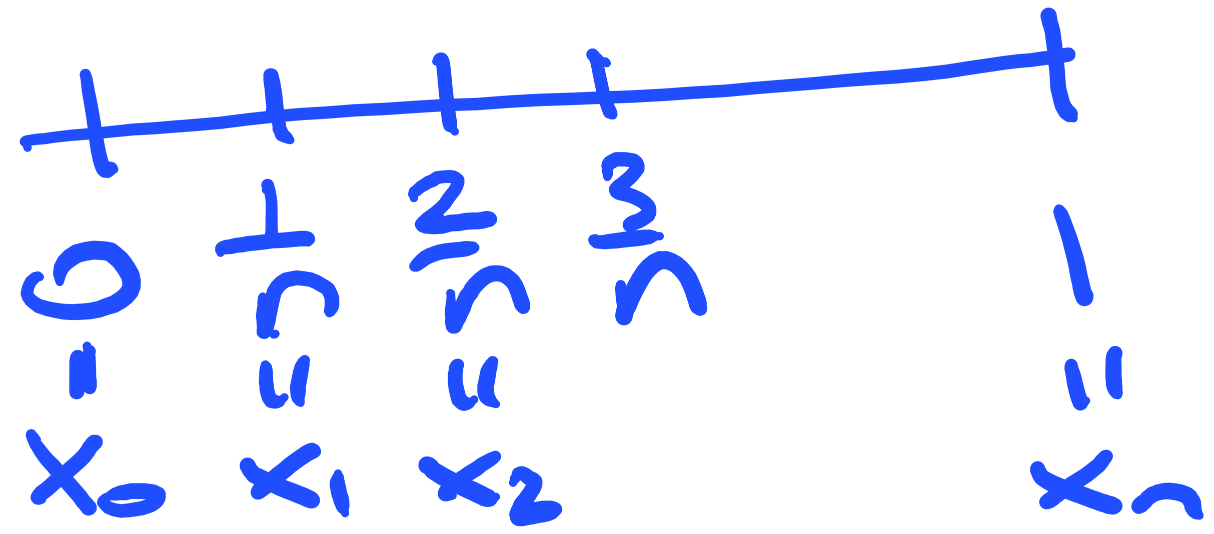
$\Delta x = \frac{b-a}{n}$. Here $a=0$ and $b=1$.

$$\therefore \Delta x_i = \frac{1-0}{n} = \frac{1}{n}.$$

$$\|P\| = \frac{1}{n}. \quad (\text{all } \Delta x_i = \tfrac{1}{n}).$$

Partition points:

$$x_i = \frac{i}{n}, \quad i = 0, 1, 2, \dots, n.$$



right end points:

$$x_i^* = \frac{i}{n}, \quad i = 1, 2, \dots, n$$

$$\text{AREA} = \lim_{n\to\infty} \sum_{i=1}^{n} f\Big(\underbrace{\tfrac{i}{n}}_{x_i^*}\Big)\, \Delta x_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3}$$

RECALL

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n^3} \sum_{i=1}^{\infty} i^2 \right)$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right)$$

$$= \frac{1}{6} \lim_{n\to\infty} \left(\left(\frac{n}{n}\right) \left(\frac{n+1}{n}\right) \frac{(2n+1)}{n} \right)$$

$$= \frac{1}{6} \lim_{n\to\infty} (1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \frac{1}{6} \lim_{n\to\infty} (1)(1)(2) = \frac{2}{6} = \frac{1}{3}$$

Exact AREA.

<u>Note:</u> $[a,b]$.

If want n equal length subinterval.

$$\Delta x_i = \frac{b-a}{n}$$

$x_0 = a$, $x_1 = a + \frac{(b-a)}{n}$, $x_2 = a + 2\frac{(b-a)}{n}$,

$\ldots$ $x_i = a + i\frac{(b-a)}{n}$, $\ldots$, $x_n = a + \frac{n(b-a)}{n} = b$.

i.e., $x_i = a + i\frac{(b-a)}{n}$, $i = 0, 1, 2, \ldots, n$.

§5.2. The DEFINITE Integral

<u>Def'n</u> If $f(x)$ is defined on $[a,b]$, let P be a partition of $[a,b]$ into subintervals $[x_{i-1}, x_i]$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_n = b.$$

Choose numbers $x_i^* \in [x_{i-1}, x_i]$

Let $\Delta x_i = x_i - x_{i-1}$.

and $\|P\| = \max\{\Delta x_i\}$

Then the "DEFINITE INTEGRAL" of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

(Reimann Sums)

if the limit exists.

NOTE: ① If the limit exists, f is call INTEGRABLE.

② $\int_a^b f(x)\,dx$ is a number.

③ $\int_a^b f(s)\,ds = \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

Many Applications

<u>Notation</u>:

$$\int_a^b$$

upper limit

integral sign

lower limit

limits of integration

$$\int_a^b \underbrace{f(x)}_{\text{Integrand.}} \, dx$$

variable of integration.

("dummy variable").

Just like the index i or k

$$\left(\sum_{i=1}^{n} i^2 = \sum_{k=1} k^2 \right)$$

Example: $\int_0^1 x^2 \, dx = \frac{1}{3}$.

We just showed this in the previous example.