§ 4.9 Antiderivatives Cont'd from last lecture.

Given the graph of f(x), draw a rough sketch of the graph of an antiderivative F(x).

Inflection points where $f'(x)=0$ (this implies $F''(x)=0$)

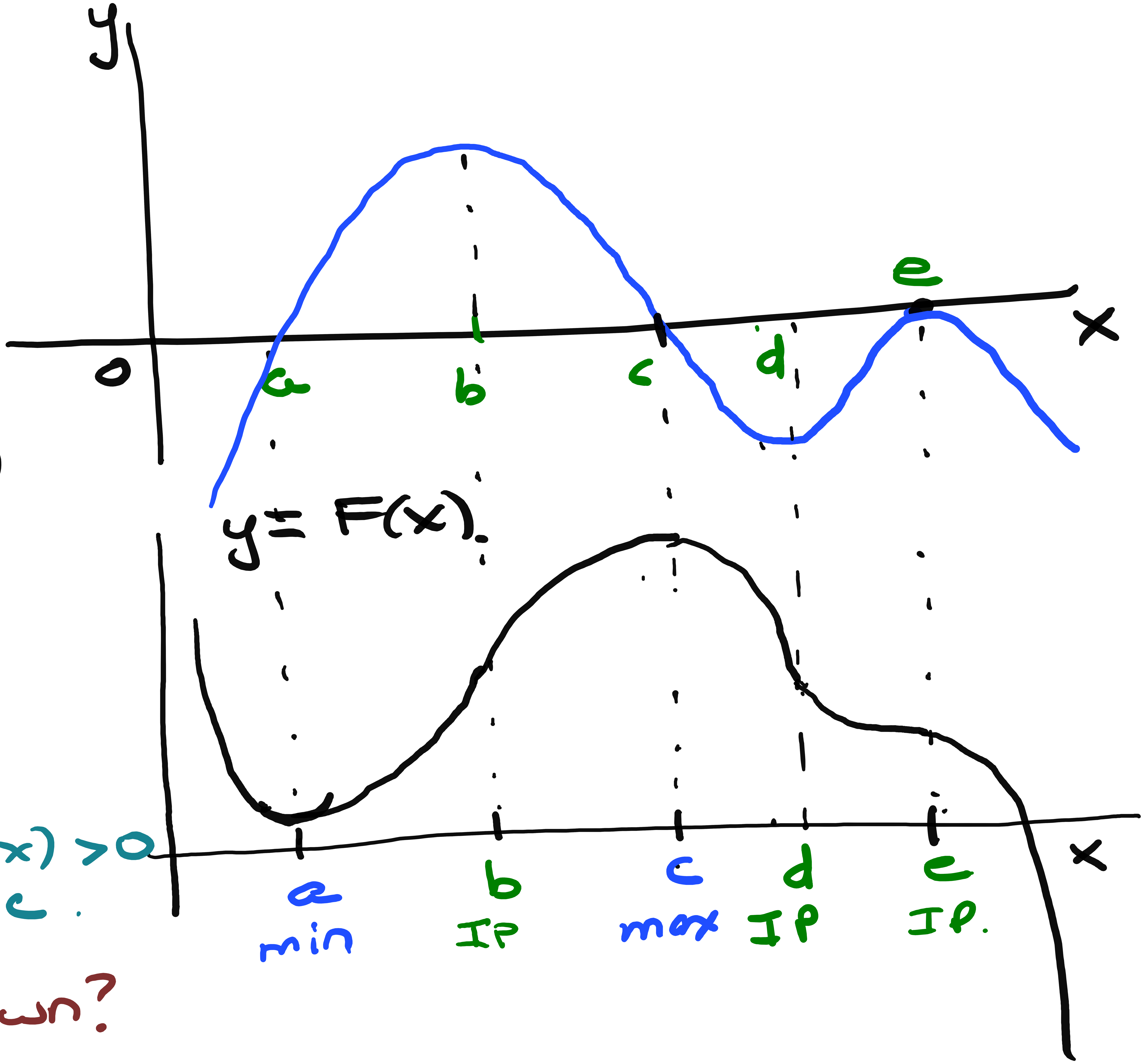
Points where $x = b, d, e$.

Local max & mins where $f(x)=0 \Rightarrow (F'(x)=0)$

But not if $f'(x)=0$ if $f(x)$ does not change sign.

min a min

max c max.



F(x) decreasing?
where $f(x) < 0$
$F'(x) = f(x)$
$x < a$ $c < x < e$ & $e < x$

increasing?
where $f(x) > 0$
$a < x < c$.

F(x) concave up? concave down?
$F''(x) = f'(x)$

$f'(x) > 0$

$\Rightarrow$ F is concave up.

$x < b$

$d < x < e$

$f'(x) < 0$

$\Rightarrow F''(x) < 0$

$\Rightarrow$ F is concave down.

$b < x < d$

$x > e$.

Example Rectilinear Motion.

A particle is accelerating at

$a(t) = 10\sin(t) + 3000t$.

Find the velocity, $v(t)$

position, $s(t)$

Given its position $\left\{ \begin{array}{l} s(0) = 0. \\ s(2\pi) = 12 \end{array} \right\}$

$v(t) = \int a(t)\,dt$ (OR $v'(t) = a(t)$)

$= \int 10\sin(t) + 3000t \; dt$

$= -10\cos(t) + 3000\frac{t^2}{2} + C$

$$s(t) = \int v(t)\,dt \qquad (s'(t) = v(t))$$

$$= \int -10\cos(t) + 1500t^2 + c \; dt$$

$$= -10\sin t + 1500\frac{t^3}{3} + Ct + D$$

C, D to be determined from $s(0) = 0$, $s(2\pi) = 12$.

$$s(0) = D = 0$$

$$s(2\pi) = -10\overset{0}{\cancel{\sin(2\pi)}} + 500(2\pi)^3 + C\,2\pi + 0 = 12.$$

$$2\pi C = 12 - (500)\,8\pi^3$$

$$C = \frac{12 - 4000\pi^3}{2\pi}$$

$$\therefore\; s(t) = -10\sin(t) + 500t^3 + \left(\frac{12 - 4000\pi^3}{2\pi}\right)t$$

$$v(t) = -10\cos(t) + 1500t^2 + \left(\frac{12 - 4000\pi^3}{2\pi}\right)t$$

(Check by differentiating.)

Appendix E. Sigma Notation

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

index.

$$= \sum_{i=m}^{m} a_i = a_m$$

$$\sum_{i=1}^{m} f(i) = f(1) + f(2) + \cdots + f(m)$$

Examples:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

$$= \frac{n(n+1)}{2}$$

Gauss:

$$\begin{array}{l} n + (n-1) + (n-2) + \cdots + 2 + 1 \\ 1 + 2 + 3 + \cdots + (n-1) + n. \\ \hline (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) \end{array}$$

$$= n(n+1)/2.$$

2. $$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n$$

3. $$\sum_{i=1}^{n} c = c \sum_{i=1}^{n} 1 = cn.$$

$c \neq 0$.

4. $$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

5. $$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Properties:

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

Pf: $$\sum_{i=1}^{n} (a_i + b_i) = (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n)$$
$$= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n)$$

$$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i .$$

$$\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i .$$

Example: $\sum_{i=1}^{n} (3 + 6i^2 + 8i^3)$

$$= 3 \sum_{i=1}^{n} 1 + 6 \sum_{i=1}^{n} i^2 + 8 \sum_{i=1}^{n} i^3$$

$$= 3n + 6 \frac{n(n+1)(2n+1)}{6} + 8 \left(\frac{n(n+1)}{2} \right)^2$$

Index Shifting

$$\sum_{i=6}^{9} a_i = a_6 + a_7 + a_8 + a_9$$

$$\sum_{i=6}^{9} a_i = \sum_{i=1}^{4} a_{i+5}$$

subtracted 5

added 5 here

Check. Are the sums the same.

Split sums.

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{5} a_i + \sum_{i=6}^{n} a_i \quad (n \geq 6).$$

More generally

$$\sum_{i=k}^{n} a_i = \sum_{i=k}^{j} a_i + \sum_{i=j+1}^{n} a_i.$$

$$\sum_{i=m}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{m-1} a_i \quad 1 < m < n.$$

Evaluating limits:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(1+\frac{i}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(1+\frac{i}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^{n} 1 + \frac{2}{n}\sum_{i=1}^{n} i + \frac{1}{n^2}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\left(n + \frac{\cancel{2}}{\cancel{n}}\left(\frac{\cancel{n}(n+1)}{\cancel{2}}\right) + \frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\cdot n + \frac{1}{n}(n+1) + \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right)$$

$$= \lim_{n\to\infty}\left(1 + \left(1+\frac{1}{n}\right) + \frac{1}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right)$$

$$= \lim_{n\to\infty} \left(1 + 1 + \frac{1}{n} + \frac{1}{6}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \right)$$

$$= \left(2 + 0 + \frac{1}{6}(1)(1+0)(2+0) \right)$$

$$= 2 + \frac{2}{6} = 2 + \frac{1}{3} = 7/3.$$

$$= 1 + 1 + \frac{1}{6} 2$$

$$= 2 + \frac{1}{3} = 7/3.$$