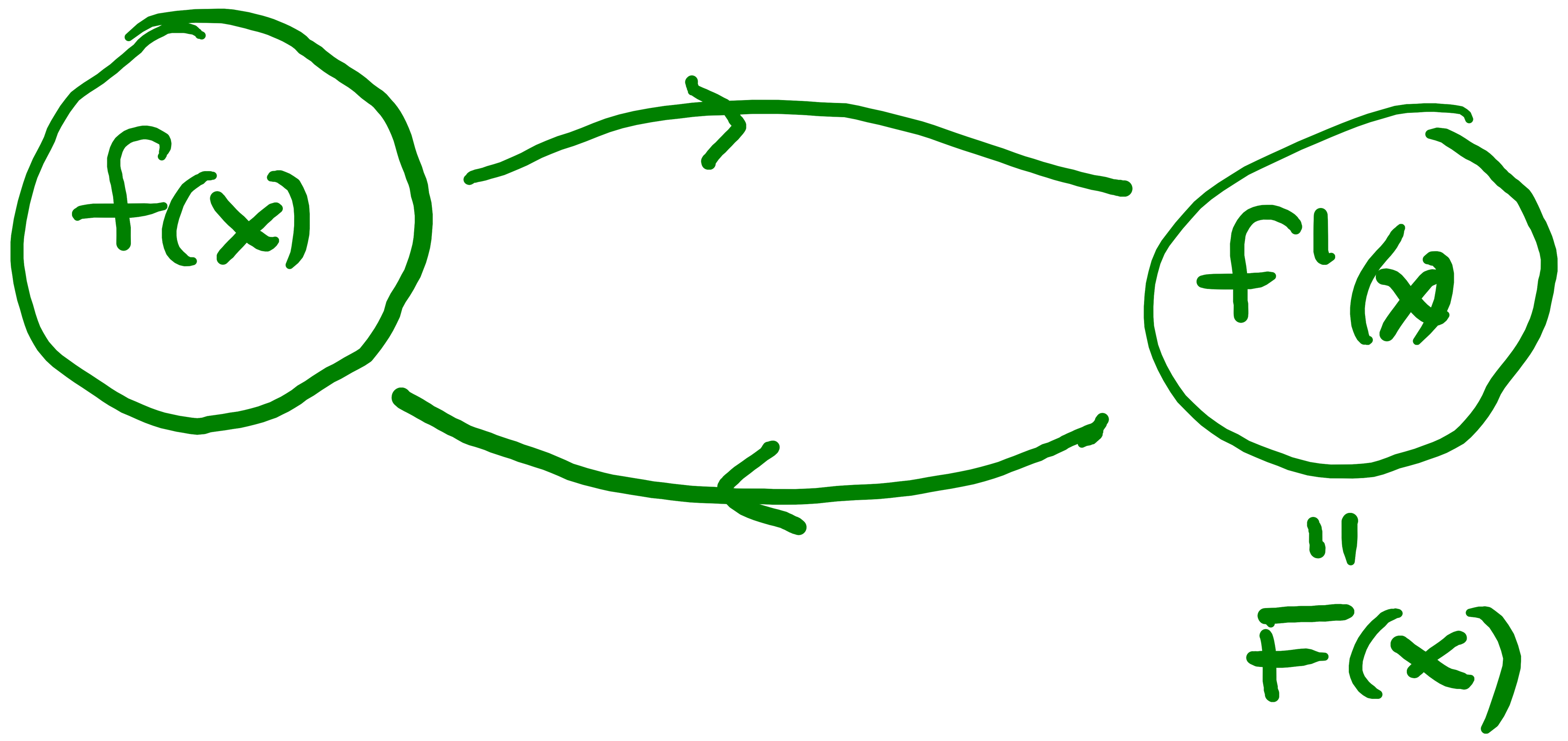
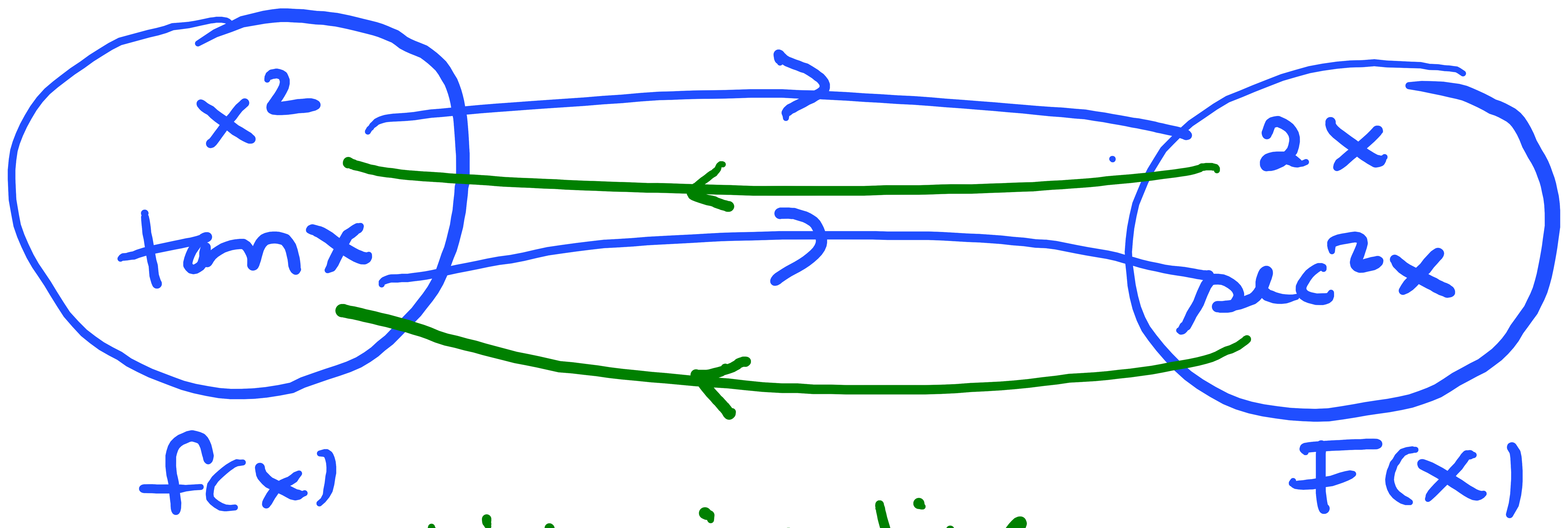
§4.9 Antiderivatives. & §5.4 Indefinite Integrals.



Given $f'(x)$ find $f(x)$.

Given $F(x)$ find $f(x)$.

Examples:



Defn: $F(x)$ is an antiderivative of $f(x)$ on an interval I if $F'(x) = f(x)$ for all x on I

Examples:

A particular antiderivative of $f(x)$.

$f(x) = \cos(x)$	$F(x) = \sin(x)$
$f(x) = -\sin(x)$	$F(x) = \cos(x)$
$f(x) = \sec^2(x)$	$F(x) = \tan(x)$

Rules: Let $F' = f$, $G' = g$

$f(x)$	$F(x)$
$c\,f(x)$	$c\,F(x)$
$f(x) \pm g(x)$	$F(x) \pm G(x)$

<u>NON UNIQUENESS</u> of antiderivatives.

If $F(x)$ is an antiderivative of $f(x)$ (i.e. $F'(x) = f(x)$), then $F(x) + C$, where C is an arbitrary constant (i.e. any constant) is an anti-derivative of $f(x)$.

<u>Pf</u>: $(F(x)+c)' = F'(x) + 0 = f(x)$.

Most general anti-derivative.

If $F'(x) = f(x)$ and $G'(x) = f(x)$ on interval I.

$\therefore F'(x) - G'(x) = 0$ on I.

$\therefore F(x) - G(x) = $ constant on I, say C.

$\therefore F(x) = G(x) + C$, C any constant.

Thm/ If F is an antiderivative of f on I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

§5.4. Indefinite Integral

Notation for anti-derivatives

"Indefinite Integral"

$$\equiv \int f(x)\,dx = F(x) + C$$

arbitrary constant

any antiderivative of f.

The INDEFINITE integral is a <u>FUNCTION</u> (<u>NOT</u> just a number).

Rules:

$$\int 0 \, dx = C$$

$$\int 1 \, dx = x + C$$

$$\int c f(x) \, dx = c \int f(x) \, dx$$

$$\int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx.$$

Examples:

$$\int \cos(x) \, dx = \sin(x) + C$$

$$\int \sin(x) \, dx = -\cos(x) + C$$

$$\int \sec^2(x) \, dx = \tan(x) + C$$

$$\int \frac{1}{x} \, dx = \ln|x| + C, \quad x \neq 0.$$

$$\int e^x \, dx = e^x + C.$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C.$$

$$\int b^x\,dx = \frac{b^x}{\ln(b)} + C. \quad (b>0)$$

$$\int \sec^2(x)\,dx = \tan(x) + C.$$

$$\int \sec x \tan(x)\,dx = \sec x + C.$$

$$\int \cosh(x)\,dx = \sinh(x) + C$$

$$\int \sinh(x)\,dx = \cosh(x) + C$$

<u>Power Rule</u>

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \underline{n \neq -1}$$

$n=-1: \int \frac{1}{x}\,dx = \ln|x| + C.$ $\cancel{\left(\text{not } \frac{x^0}{0}\right)}$

Example: $\int \sqrt{x}\,dx = \int x^{1/2}\,dx$

$$= \frac{x^{1/2+1}}{1/2+1} + C$$

$$= \frac{2x^{3/2}}{3} + C.$$

Emphasize: Check by differentiating.

Evaluate: $\int \frac{3}{1+x^2} + \sec^2 x \; dx$

$$= 3\int \frac{1}{1+x^2}\,dx + \int \sec^2 x\,dx$$

$$= 3\tan^{-1}(x) + \tan(x) + C$$

Check by differentiating.

Example: Find $f(x)$ if $f'(x) = \dfrac{x+1}{\sqrt{x}}$ and $f(1)=5$.

Sol'n: $f'(x) = \dfrac{x}{\sqrt{x}} + \dfrac{1}{\sqrt{x}}$

$$\left\{ \frac{x}{\sqrt{x}} = \frac{(\sqrt{x})^2}{\sqrt{x}} = \sqrt{x} \quad \text{OR} \quad \frac{x}{\sqrt{x}} = x^1 x^{-1/2} = x^{\frac{1}{2}} \right\} \longrightarrow \; = x^{1/2} + x^{-1/2}$$

$$f(x) = \int x^{1/2} + x^{-1/2}\, dx$$

$$= \frac{2}{3}x^{3/2} + \frac{x^{-1/2+1}}{-1/2+1} + C.$$

$$f(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} + C.$$

C to be determined from $f(1)=5$.

$$5 = f(1) = \frac{2}{3}(1)^{3/2} + 2(1)^{1/2} + C$$

$$5 = \frac{2}{3} + 2 + C \implies C = 2\tfrac{1}{3} \text{ or } C = 7/3.$$

Example: Find $f(x)$ if $f'''(x) = \cos(x)$

$$\begin{cases} f(0) = 1 \\ f'(0) = 4 \\ f''(0) = 3. \end{cases}$$

Sol'n. $f'''(x) = \cos(x)$

$$f''(x) = \int f'''(x)\,dx$$

$$= \int \cos(x)\,dx = \sin(x) + C$$

C to be determined.

$$f'(x) = \int f''(x)\,dx$$

$$= \int \sin(x) + C \quad dx$$

$$= -\cos(x) + Cx + D$$

C, D to be determined.

$$f(x) = \int f'(x)\,dx$$

$$= \int -\cos(x) + Cx + D \, dx$$

$$f(x) = -\sin(x) + C\frac{x^2}{2} + Dx + E$$

C, D, E to be determined.

$$f''(x) = \sin(x) + C$$

given: $f''(0) = 3 = \underbrace{\cancel{\sin(0)}}_{0} + C \Rightarrow \underline{C = 3}.$

$$f'(x) = -\cos(x) + \underset{C=3}{3}x + D$$

given: $f'(0) = 4 = -\cos(0) + 3(0) + D$

$$4 = -1 + D \Rightarrow \underline{D = 5}.$$

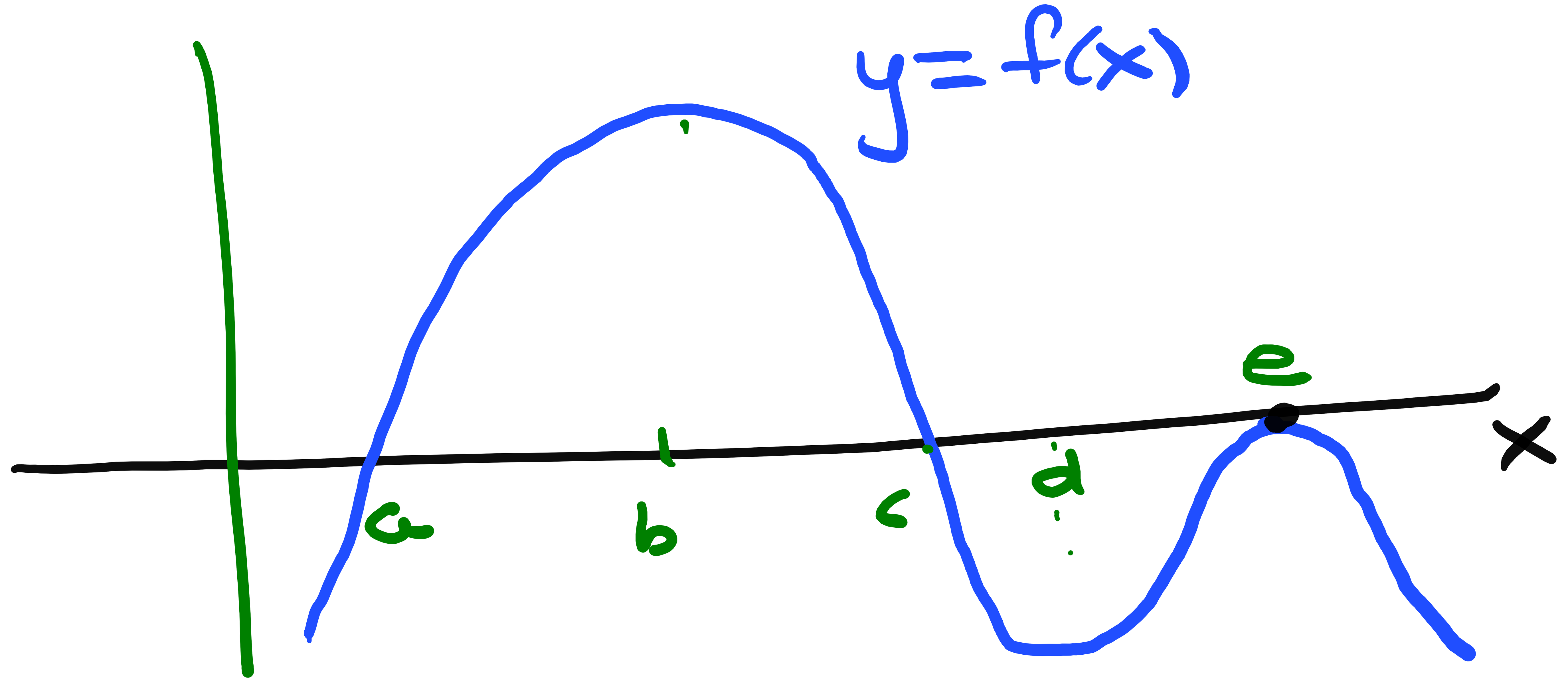
$$f(x) = -\sin(x) + \overset{C}{3}\frac{x^2}{2} + \overset{D}{5}x + E$$

$$1 = f(0) = -\sin(0) + \frac{3}{2}0^2 + 5(0) + E.$$

$$\underline{E = 1.}$$

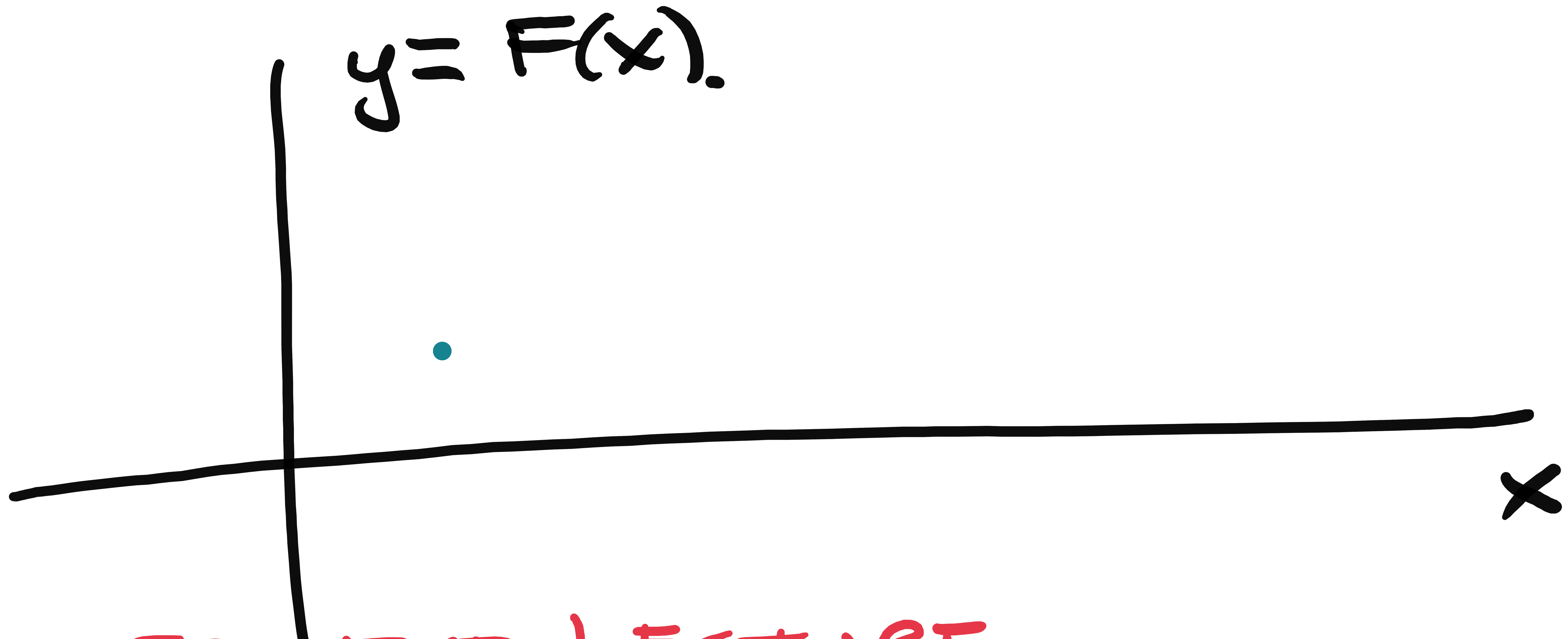
$$\therefore \quad f(x) = -\sin(x) + \frac{3}{2}x^2 + 5x + 1.$$

Given graph of $f(x)$
Draw a rough sketch of an antiderivative $F(x)$.



Inflection points where $f'(x) = 0$ (this implies $F''(x) = 0$)

Points where $x = b, d, e$.



local max + mins

$f(x) = 0$ ($\Rightarrow F'(x) = 0$)

but $f'(x) \neq 0$.

a + c

(not e).

TO BE CONTINUED NEXT LECTURE.