§4.7 Optimization.

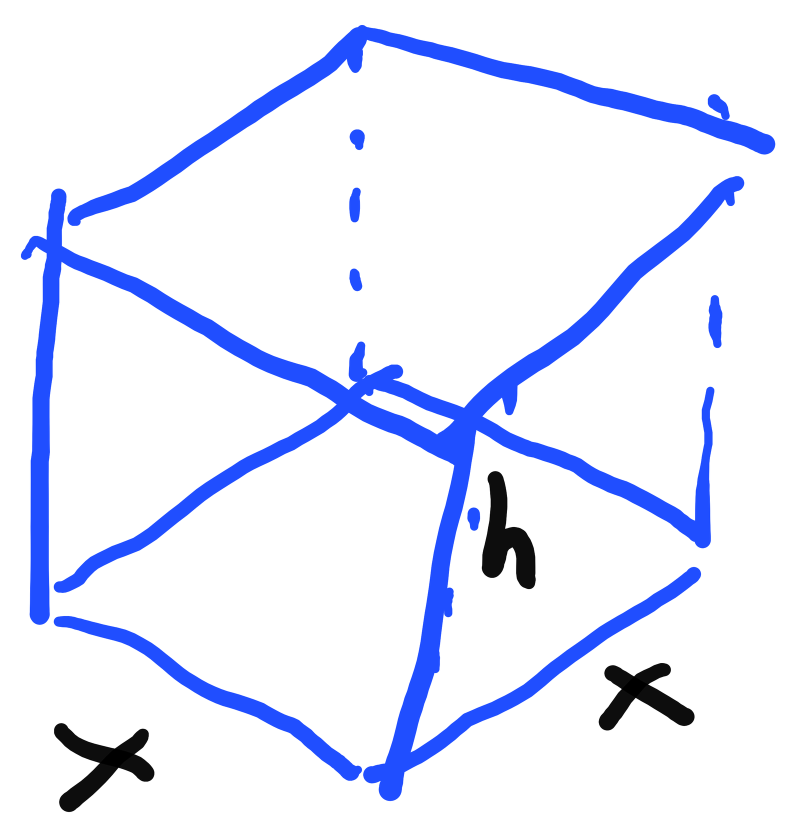
Example. A box with a square base and an open top has a volume of 32,000 cm^3. Find the dimensions of the box that minimizes the amount of material used.

1) Read the question carefully.

What do we know?

What is unknown that we need to find out?

2) Draw a picture & identify what you know.



$$\text{Volume} = x^2 h = 32{,}000$$

3). Introduce notation:

Let x denote the length = width of the base (in cm) of the box.

h denote the height of the box (in cm.)

4) Express the Surface Area, S (the quantity you wish to minimize) in terms of the variables.

$$S(x, h) = \text{area of base} + 4 \times (\text{area of each side})$$

$$= x^2 + 4xh.$$

Find x, h to minimize S

where $V = x^2 h = 32{,}000 \text{ cm}^3$.

Method I.

5) Express function we wish to optimize, $S(x,h)$ (in the case minimize) in terms of only one variable. (i.e. solve for h as a function of x.)

$$V = x^2 h = 32{,}000$$

$$\Rightarrow h = 32{,}000/x^2$$

$$\therefore S(x,h) = S(x) = x^2 + 4x\left(\frac{32{,}000}{x^2}\right),$$ (a function of only 1 variable)

$$S(x) = x^2 + \frac{128{,}000}{x}$$

6). Find absolute min. of $S(x)$ (if possible).

Critical numbers.

Domain: $x > 0$, $h > 0$.

$$S'(x) = 2x - \frac{128{,}000}{x^2}.$$

Solve $S'(x) = 0$ for x.

$$2x - \frac{128{,}000}{x^2} = 0$$

$$x^3 = 64{,}000$$

$$x = 40 \text{ cm}.$$

and $\therefore$ $h = \frac{32{,}000}{x^2} = \frac{32{,}000}{(40)^2} = 20$ cm.

Is this a local max, min or inflection point?

(2nd deriv. test) $S''(x) = 2 + \frac{(2)(128{,}000)}{x^3} > 0$ for all $x > 0$

$\Rightarrow$ S has a local min at $x = 40$

Since there is only 1 critical number, this implies that S has an absolute min at $x = 40$ cm and $h = 20$ cm.

7) Read the question again to make sure you are providing the answer to what was asked.

The dimensions of the box with the min material required is the box 40 cm × 40 cm and the height 20 cm.

Alternative method (from step 5 above).

Know: $\begin{cases} V(x,h) = x^2 h = 32{,}000 \text{ cm}^3 \\ S(x,h) = x^2 + 4xh. \end{cases}$

Insted of solving for h as a function of x to get S(x) think of $h = h(x)$ and use implicit differentiation.

(1) $\frac{dS}{dx} = 2x + 4h + 4xh' = 0$ To find critical numbers.

(2) $\frac{dV}{dx} = 2xh + x^2 h' = \frac{d}{dx} 32{,}000 = 0$

$\Rightarrow x(2h + xh') = 0 \Rightarrow 2h + xh' = 0$ (since $x \neq 0$)

$\therefore$

$(1)'$ $x + 2h + 2xh' = 0$ from (1)/2.

$(2)'$ $2h + xh' = 0.$ from (2).

Eliminate h' by considering
$2 \cdot$ eqn $(2)' -$ eqn. $(1)'$

$$2(2h + xh') - (x + 2h + 2xh') = 0.$$

$$4h - x - 2h = 0$$

$$\Rightarrow \boxed{2h = x}$$

Since $V = x^2 h = 32{,}000.$

$x = 2h \Rightarrow (2h)^2 h = 32{,}000$

$$4h^3 = 32{,}000$$

$$h^3 = 8{,}000$$

$$\boxed{h = 20 \text{ cm.}}$$

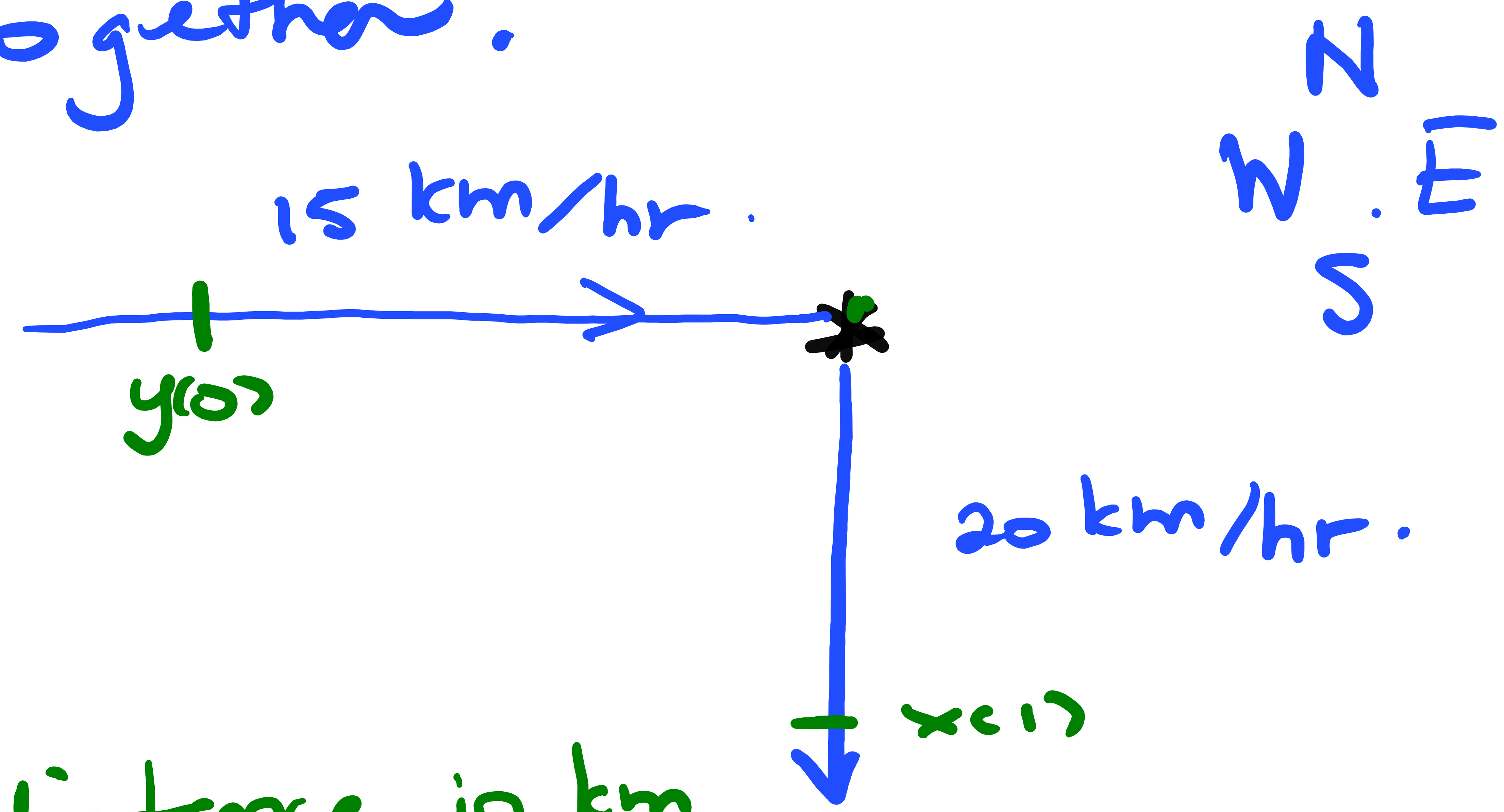
$x = 2h \Rightarrow \boxed{x = 40 \text{ cm.}}$

As before, you would need to think about this is where there is a local min, local max, or an inflection point etc.

Example: Boat 1 leaves a dock at 2pm and travels due SOUTH at a speed of 20 km/hr.

Another boat has been heading to the same dock traveling due EAST at 15 km/hr and arrives at that dock at 3:00 pm.

At what time were the boats the closest together?



Let t denote time in hours.

Let $t=0$ be 2:00 pm

$t=1$ be 3:00 pm.

Let $x(t)$ denote the distance in km. of boat 1 from the dock at time t

$\therefore \quad x(0) = 0$ km (at dock at 2:00 pm)

$x(1) = 20$ km. ($t=1$ is 3:00 pm)

$x(t) = 20t$ km.

Let $y(t)$ denote the distance of boat 2 from the dock.

$y(0) = 15$ (It takes 1hr to get to the dock at 15 km/hr)

$y(1) = 0$ (It reaches the dock at 3:00 pm)

$y(t) = 15 - 15t$ (Distance from the dock, $0 \le t \le 1$

Minimize the distance D, between the 2 boats for $0 \le t \le 1$. (Note: $D(0) = 15$km, $D(1) = 20$ km).

$$D(t) = \sqrt{(x(t))^2 + (y(t))^2} \qquad 0 \le t \le 1.$$

$$= \sqrt{(20t)^2 + (15-15t)^2}$$

Note the optimal t for $D(t)$ is the same as for $(D(t))^2$ since the value of t that makes $D(t)$ the smallest is the same value of t that makes $(D(t))^2$ the smallest.

Let $\tilde{D}(t) = (D(t))^2$

Find t the minimizes: $\tilde{D}(t) = (20t)^2 + (15-15t)^2$.

Find the critical numbers:

$$\tilde{D}'(t) = 2(20t)20 - 2(15-15t)15$$

$$\boxed{\tilde{D}'(t) = -450 + 1250t}.$$

Find t so that $\tilde{D}'(t) = 0$.

$$-450 + 1250t = 0 \Rightarrow t = 9/25.$$

Check if this is where $\tilde{D}(t)$ gives a local max, min, or inflect point.

e.g. Using the 1st derivative test:

$\tilde{D}'(t) < 0$ if $0 < t < 9/25$

& $\tilde{D}'(t) > 0$ if $t > 9/25$.

at $t = 9/25$; $\tilde{D}(t)$ and hence $(\tilde{D}^2(t)) = D(t)$, are local mins.

Since it is the only critical number in $(0,1)$, it is the absolute min. However, can compare with the distances at the end points. (i.e. at 2:00 pm they are 15 km apart
3:00 pm they are 20 km. apart)

At $t = 9/25$ they are

$$D(t) = \sqrt{(20(9/5))^2 + (15-15(9/5))^2} = \quad \text{km}$$

apart.

Answer the question asked:

The two boats are the closest when $t = 9/25$.

This is $2^{:00}_{pm} + (9/25)\,hrs$.

$$= 2^{:00}_{pm} + (9/25)\,hrs \cdot \frac{60\,min}{1\,hr}$$

$$= 2^{:00}_{pm} + 21\tfrac{3}{5}\,min.$$

$\frac{3}{5}\,min = \frac{3 \cdot 60\,sec}{5}$
$= 36\,sec.$

$\therefore$ They are closest together at $2{:}21{:}36$ pm.

(i.e. 21 minutes and 36 seconds after 2pm)