§ 4.4 cont'd

4. **Powers** What if $(f(x))^{g(x)}$

where "0^0" OR "∞^0" OR "$1^{\pm\infty}$"

Convert to a product using logarithms.

Example: $\lim_{x\to\infty} \left(\underbrace{1+\frac{a}{x}}\right)^{bx}$, $a, b \in \mathbb{R}$, $a\neq 0, b\neq 0$

$f(x) \to 1$ as $x \to \infty$

$g(x) = bx \to \infty$ as $x \to \infty$

"1^{∞}"

Let $y = \left(1+\frac{a}{x}\right)^{bx}$

$\ln y = bx \ln\left(1+\frac{a}{x}\right)$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} bx \ln\left(1+\frac{a}{x}\right)$ "$\infty \cdot 0$"

$$\left.\begin{array}{l} x \to \infty \\ \ln\left(1+\frac{a}{x}\right) \to 0 \end{array}\right\} \text{ as } x \to \infty.$$

$$\lim_{x\to\infty} \ln y$$

$$= b \lim_{x\to\infty} \frac{\ln\left(1+\frac{a}{x}\right)}{\left(\frac{1}{x}\right)} \qquad \text{"}\frac{0}{0}\text{"}$$

$$\underset{\text{l'HR}}{=} b \lim_{x\to\infty} \frac{\frac{1}{1+a/x}\left(-\frac{a}{x^2}\right)}{\left(-1/x^2\right)}$$

simplify

$$= ab \lim_{x\to\infty} \frac{1}{1+\frac{a}{x}} = ab$$

$$\lim_{x\to\infty} \ln\left(f(x)\right)^{g(x)} = ab$$

$$\lim_{x\to\infty} \left(f(x)\right)^{g(x)} = \lim_{x\to\infty} e^{\ln f(x)^{g(x)}}$$

$$\lim_{x\to\infty} \left(f(x)\right)^{g(x)} = \lim_{x\to\infty} e^{\ln f(x)^{g(x)}} = \underset{\sim}{e^{ab}}.$$

Indeterminate forms:

"$\frac{0}{0}$"   "$\frac{\pm\infty}{\pm\infty}$"

"$0\cdot\infty$"   "$0\cdot(-\infty)$"

"$\infty-\infty$", "$-\infty+\infty$"

"1^{∞}" $(1-\text{small})^{\infty}\to 0$

"$1^{-\infty}$" $(1+\text{small})^{\infty}\to\infty$

$\oplus$

"0^0"

$$\begin{cases} x^0 = 1 & \text{if } x\neq 0 \\ 0^x = 0 & \text{if } x\neq 0. \end{cases}$$

"∞^0"   Example: $\lim_{x\to\infty}(\ln x)^{1/x}$

$\ln y = \frac{1}{x}\ln(\ln(x))$ "$\frac{\infty}{\infty}$"

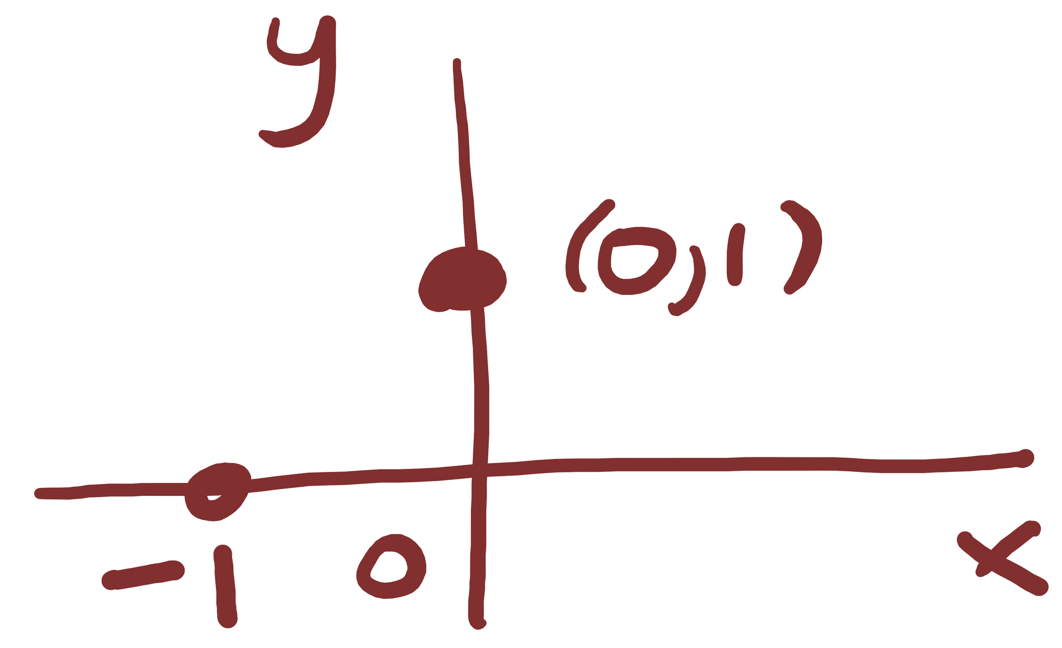
$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\frac{1}{\ln(x)}\cdot\frac{1}{x}}{1} = 0$$

$$\therefore \lim(\ln x)^{1/x} = e^{\lim_{x\to\infty}\ln y} = e^0 = \underline{1}.$$

§4.5 Curve Sketching.

Example: $f(x) = \dfrac{x+1}{\sqrt{x^2+1}} = y$

ⓐ $\text{dom}(f) = \mathbb{R}$.



ⓑ x, y intercepts.

y-intercept. is where $x=0$

$f(0) = 1$. point $(0, 1)$

x-intercept is where $y=0$.

$f(x)=0$ point $(-1,0)$

ⓒ symmetry (if f is even?

odd. ?

periodic?)

Not in this case.

ⓓ asymptotes:

- continuous so no vertical asymptotes.

What about horizontal asymptotes.

Look for whether $\lim_{x\to\infty} f(x) = L$ a finite #

and $\lim_{x\to-\infty} f(x) = M$ " " "

$$\lim_{x\to\infty} \frac{x+1}{\sqrt{x^2+1}} \qquad \text{"}\frac{\infty}{\infty}\text{"}$$

$$\overset{l'HR}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{2}(x^2+1)^{-1/2}(2x)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} \qquad \text{"}\frac{\infty}{\infty}\text{"}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2}(x^2+1)^{-1/2}\, 2x}{1}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$$

Homework: Try applying l'H.R. 2 more times.

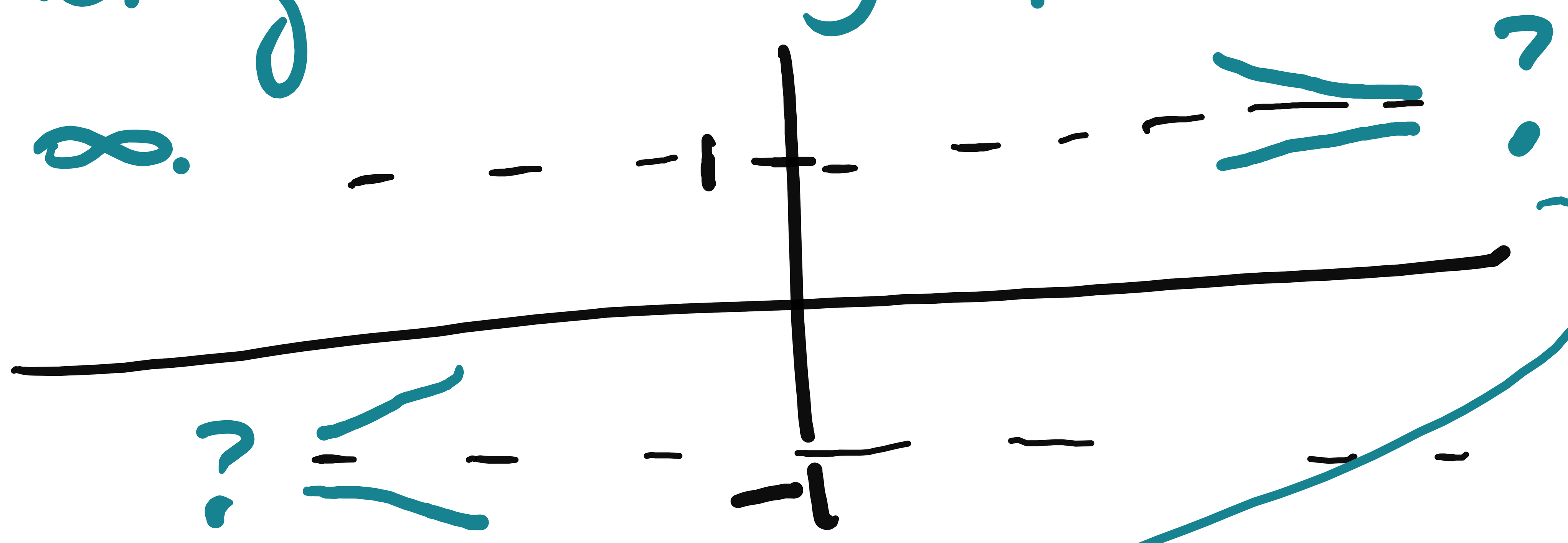
You will see that you are going in circles, and getting nowhere.

Instead:

$$\lim_{x\to\infty} \frac{(x+1)/x}{(\sqrt{x^2+1})/x} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}} = \frac{1}{1} = 1.$$

$\therefore$ there is a horizontal asymptote as $x \to \infty$.



Need 1st derivative info to distinguish which it is.

BE CAREFUL:

$$\lim_{x\to-\infty} \frac{x+1}{\sqrt{x^2+1}}, \quad x<0$$

$$= \lim_{x \to -\infty} \frac{(x+1)/x}{\sqrt{x^2+1}/(-(-x))}$$

$$= \lim_{x \to -\infty} \frac{1 + \frac{1}{x}}{-\sqrt{\frac{x^2}{(-x)^2} + \frac{1}{(-x)^2}}}$$

$$= \lim_{x \to -\infty} - \frac{\left(1 + \frac{1}{x}\right)}{\sqrt{1 + \frac{1}{(-x)^2}}} = \underline{-1.}$$

(e) Find where $f(x)$ ↑ and where it is ↓, using $f'(x)$.

$$f(x) = \frac{x+1}{\sqrt{x^2+1}} \rightarrow f'(x) = \quad \text{HW}$$

$$\vdots$$

$$= \frac{1-x}{(x^2+1)^{3/2}}$$

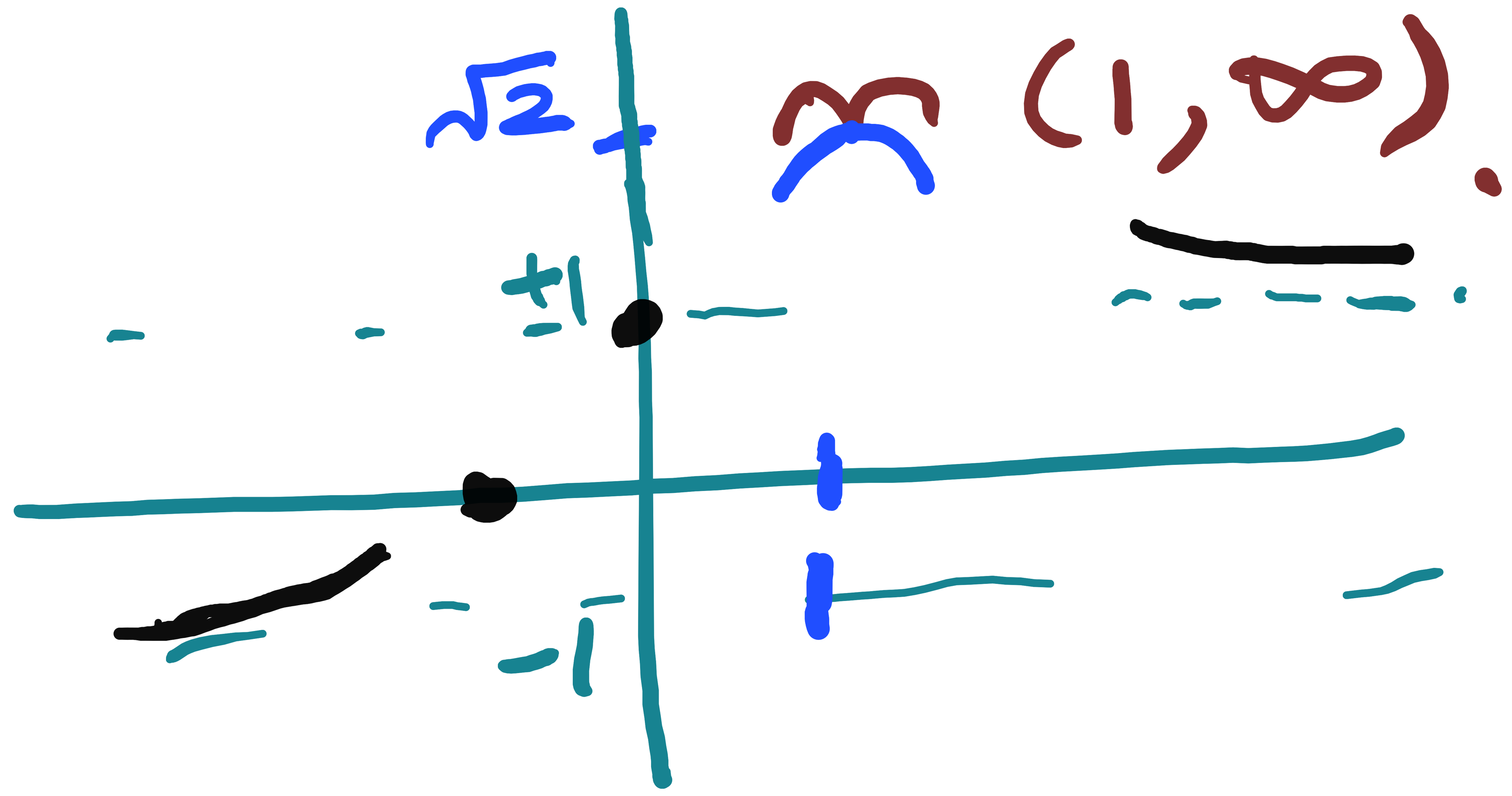
$f'(x)$ exists everywhere.

$f'(x) = 0$ where $1 - x = 0 \Rightarrow x = 1$.

$f'(x) > 0$ if $x < 1$ $\therefore$ $f \uparrow$ on $(-\infty, 1)$

$f'(x) < 0$ if $x > 1$ $\therefore$ $f \downarrow$ on $(1, \infty)$.

(Now you know how the horizontal asymptotes are approached.)



(f) local max + min.

$f'(x) = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$.

1st deriv. test. $f(1) = \frac{2}{\sqrt{2}} = \sqrt{2}$.

$f' > 0$ $x < 1$ & $f'(x) < 0$ $x > 1$
$\Rightarrow$ f has a local max at $x = 1$.
with value $f(1) = \sqrt{2}$.

9 Concavity & pts. of inflection.

$$f''(x) = \vdots \quad HW$$

$$= \frac{2x^2 - 3x - 1}{(x^2+1)^{5/2}}$$

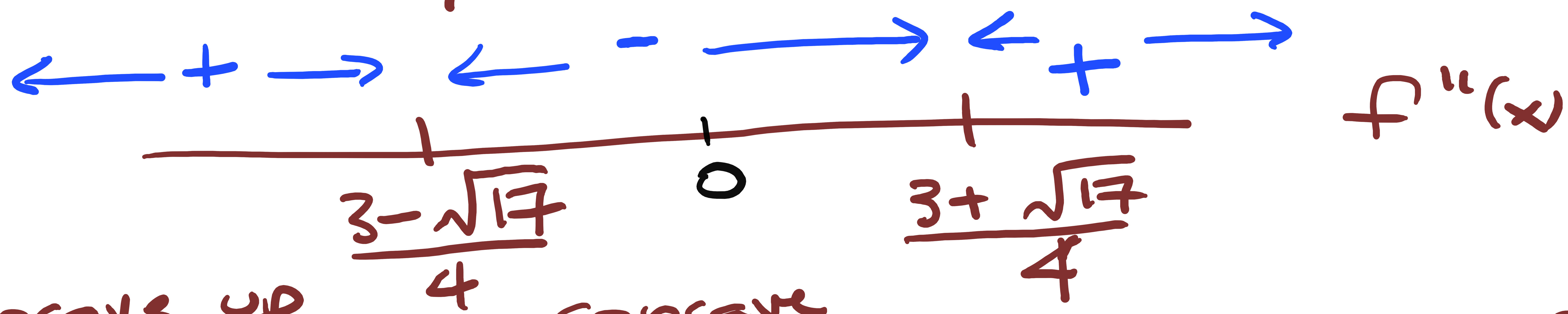
dnom + always.

Roots of numerator using the quadratic formula:

$$2x^2 - 3x - 1 = 0$$

$$x = \frac{3 \pm \sqrt{9 - 4(2)(-1)}}{(2)(2)}$$

$$= \frac{3 \pm \sqrt{17}}{4}$$



concave up $\left(-\infty, \frac{3-\sqrt{17}}{4}\right)$

concave down $\left(\frac{3-\sqrt{17}}{4}, \frac{3+\sqrt{17}}{4}\right)$

concave up $\left(\frac{3+\sqrt{17}}{4}, \infty\right)$