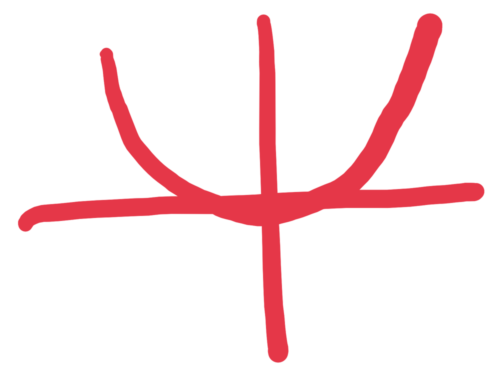
§4.3 con'td.

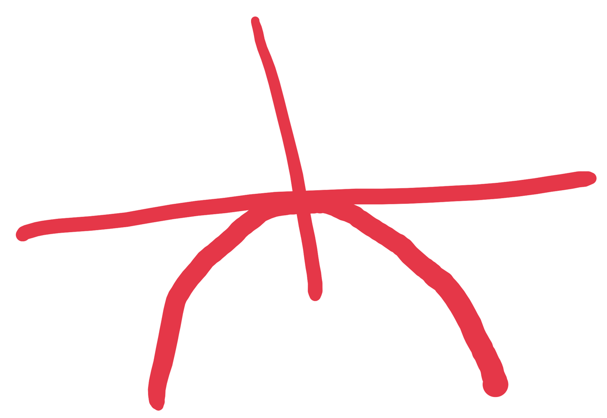
2nd Derivative Test for local max & local mins.

Assume f'' is continuous near c.
(i.e. in an open interval containing c)

(a) If $f'(c)=0$ and $f''(c)>0$, then f has a local min at $x=c$

(think $f(x)=x^2$
$f'(0)=0$
$f''(0)=2>0$
$f'(x)>0$ near 0.

)

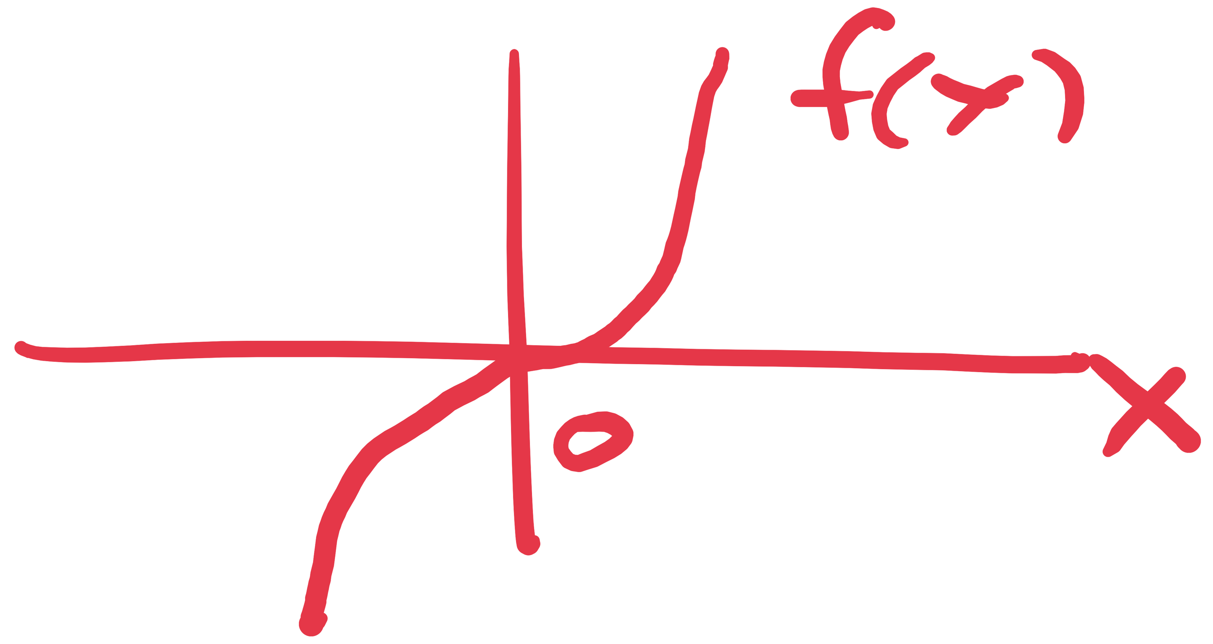
(b). If $f'(c)=0$ and $f''(c)<0$, then f has a local max at $x=c$.

(think $f(x)=-x^2$

)
$f'(x)=-2x$
$f'(0)=0$

$f''(x) = -2 < 0$

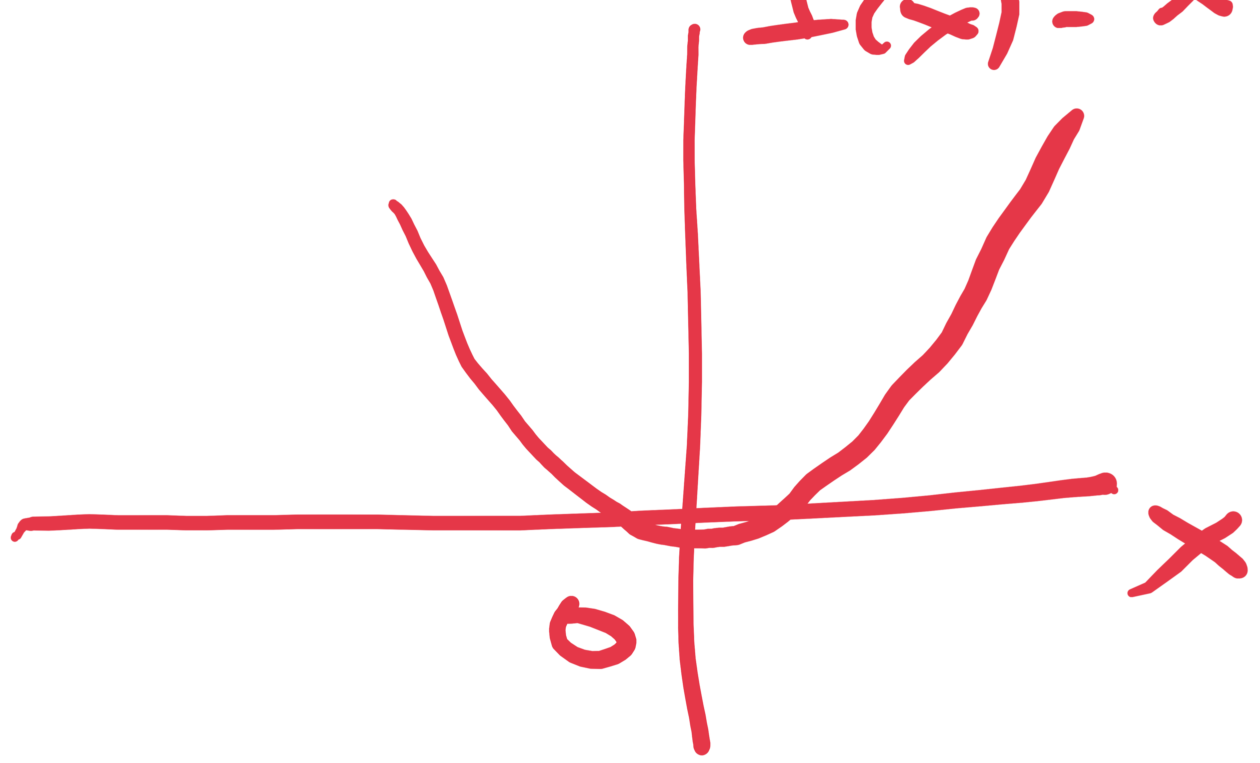
(C) If $f'(c) = 0$ and $f''(c) = 0$, ANYTHING GOES.
i.e. test is inconclusive.

Eg $f(x) = x^3$

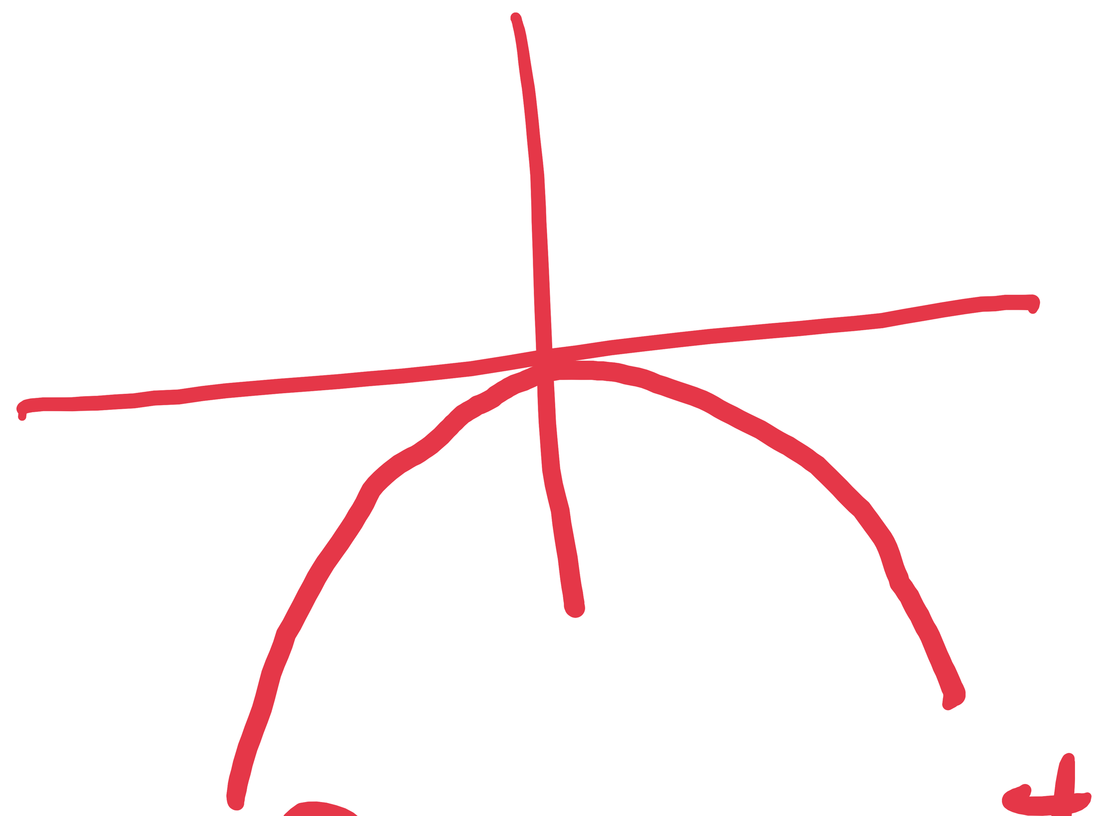


$x = 0$ is an inflection pt.

$f(x) = x^4$



$f(x) = x^4$
local min at $x = 0$



$f(x) = -x^4$
local max at $x = 0$

$$\left.\begin{array}{l} f'(0) = 0 \\ f''(0) = 0 \end{array}\right\}$$ NO information from 2nd deriv. test.

USE the 1st derivative test.

§4.4. Indeterminate Forms & L'Hospital's Rule.

Question: What if we want to evaluate

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

when $\lim_{x \to a} f(x) = 0$ & $\lim_{x \to a} g(x) = 0.$?

("a" could be finite, or $+\infty$ or $-\infty$).

I sometimes can use algebraic simplification.

e.g. $$\lim_{x \to 2} \frac{x^2 - 2x}{x^2 - 4} = \lim_{x \to 2} \frac{x \cancel{(x-2)}}{\cancel{(x-2)}(x+2)} = \lim_{x \to 2} \frac{x}{x+2} = \frac{1}{2}.$$

But what about

$$\lim_{x \to 0} \frac{\tan x}{\tanh(x)} \quad \text{OR} \quad \lim_{x \to \infty} \frac{\ln x}{x} \ ?$$

1. Quotients $f(x)/g(x)$.

L'Hospital's Rule.

Assume that f & g are diff'ble and $g'(x) \neq 0$ on an open interval containing a except possibly at a.

Assume $\lim_{x\to a} f(x) = 0$ & $\lim_{x\to a} g(x) = 0$.

(OR. $\lim_{x\to a} f(x) = \pm\infty$ & $\lim_{x\to a} g(x) = \pm\infty$)

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if this limit exists or is $+\infty$ or $-\infty$.

NOTE: "a" can also be $+\infty$ or $-\infty$. (can also be for a^+ or a^-.).

Proof (When $f(a) = g(a) = 0$ and f', g' are continuous on I containing a and $g'(a) \neq 0$).

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} \qquad (f(a) = g(a) = 0)$$

$$= \lim_{x \to a} \frac{(f(x) - f(a))/(x-a)}{(g(x) - g(a))/(x-a)}$$

$$= \frac{\lim_{x \to a} \frac{f(x) - f(a)}{x-a}}{\lim_{x \to a} \frac{g(x) - g(a)}{x-a}}$$

$$= f'(a)/g'(a).$$

$\not\propto$.

Example. 1. $\lim_{x\to 0} \frac{\tan x}{\tanh(x)}$ $\left(\text{"}\frac{0}{0}\text{"}\right)$.

$$\underset{\text{l'H.R.}}{=} \lim_{x\to 0} \frac{\frac{d\tan x}{dx}}{\frac{d\tanh(x)}{dx}}$$

$$= \lim_{x\to 0} \frac{\sec^2 x}{\operatorname{sech}^2 x} = \frac{1}{1} = \underline{1}.$$

Example 2.
(repeated use).

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} \qquad \text{"}\frac{0}{0}\text{"}$$

$$\underset{\text{l'HR}}{=} \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \qquad \text{"}\frac{0}{0}\text{" again.}$$

repeat using l'HR.

$$\underset{\text{l'H.R}}{=} \lim_{x \to 0} \frac{2\sec^2 x \tan x}{6x} \qquad \text{"}\frac{0}{0}\text{" again!}$$

$$\underset{\text{l'H.R.}}{=} \lim_{x \to 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x}{6}$$

$$= \frac{0+2}{6} = \frac{1}{3}. \qquad \underline{\text{this exists.}}$$

Example $\displaystyle \lim_{x \to \infty} \frac{\ln x}{x} \qquad \text{"}\frac{\infty}{\infty}\text{"}$

$$\underset{\text{l'HR}}{=} \lim_{x \to \infty} \frac{1/x}{1} = 0$$

BEWARE: Only use L'Hospital's rule if it is an indeterminate form.

② **Products** $\lim_{x\to a} f(x)g(x)$

What if $\lim_{x\to a} f(x) = 0$ & $\lim_{x\to a} g(x) = \infty$ or $-\infty$.

"$0\cdot\infty$" or "$0\cdot(-\infty)$".

Convert $fg = \dfrac{f}{1/g}$ or $fg = \dfrac{g}{1/f}$

Example: $\lim_{x\to 0^+} \sqrt{x}\ln x$

$\lim_{x\to 0^+} \sqrt{x} = 0$

$\lim_{x\to 0^+} \ln x = -\infty$.

("$0\cdot(-\infty)$".

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} \quad \text{"}\frac{-\infty}{\infty}\text{"}$$

$$\underset{\text{l'HR}}{=} \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} \quad \text{simplify}$$

$$= \lim_{x\to 0^+} -2\sqrt{x} = 0.$$

③ Difference: $\lim_{x\to a} (f(x) - g(x))$.

What if $\lim_{x\to a} f(x) = \infty$ & $\lim_{x\to a} g(x) = \infty$.

$= -\infty$ & $-\infty$.

Convert the difference to a quotient.

Use: common denom.

OR rationalize.

or common factor.

Example: $\lim_{x\to 0} \left(\frac{1}{x} - \csc x\right)$

$= \lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$ "$\infty - \infty$".

common denom.

$$= \lim_{x\to 0} \frac{\sin x - x}{x\sin x} \quad \text{"}\tfrac{0}{0}\text{"}.$$

$$\underset{\text{l'H.R.}}{=} \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} \quad \text{"}\tfrac{0}{0}\text{"}$$

repeat

$$\underset{\text{l'HR.}}{=} \lim_{x\to 0} \frac{-\sin x}{\cos x + \cos x - x\sin x}$$

$$= \lim_{x\to 0} \frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0.$$

(F) Powers. What if $\lim_{x\to a} \left(f(x)\right)^{g(x)}$

where "0^0" OR "∞^0" OR "$1^{\pm\infty}$"

Convert to a product taking logarithms.

Cont'd tomorrow.

Good Luck Tonight