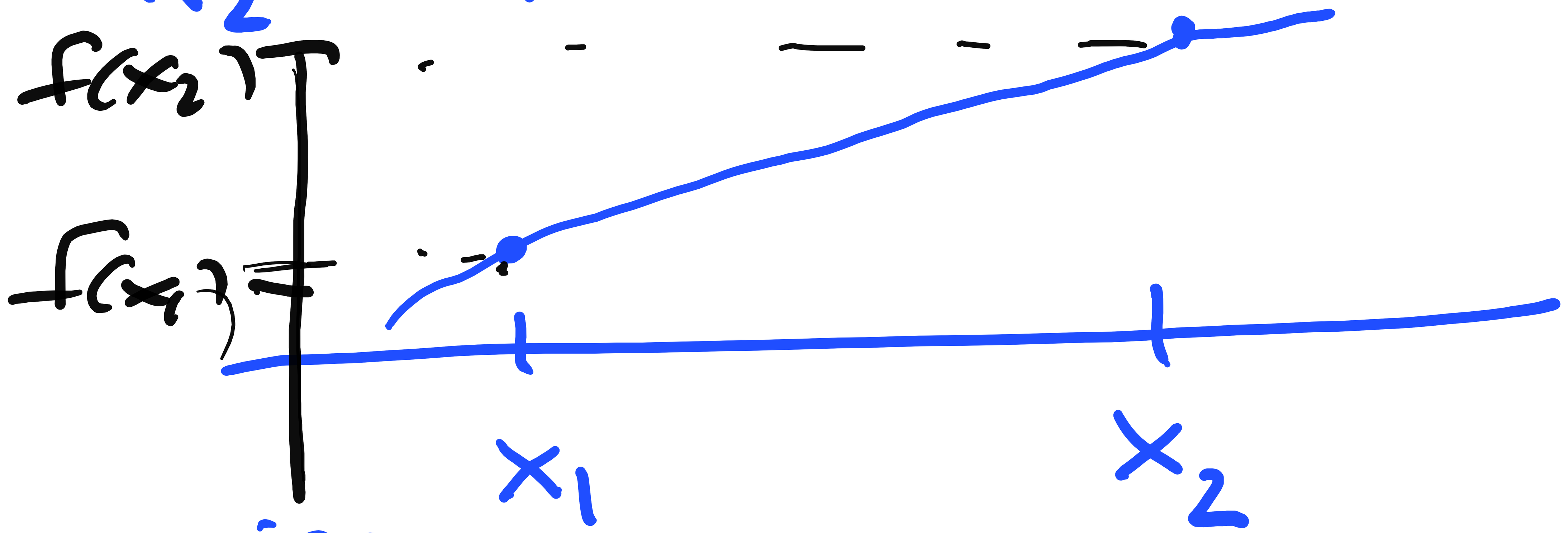
§4.3. Derivatives — Slope of a Graph.

$f'(c) = k \Rightarrow$ tangent to the graph of f at $x=c$ has slope k.

Def'n f is increasing ($\uparrow$) on an interval I

if for any $x_1, x_2 \in I$,

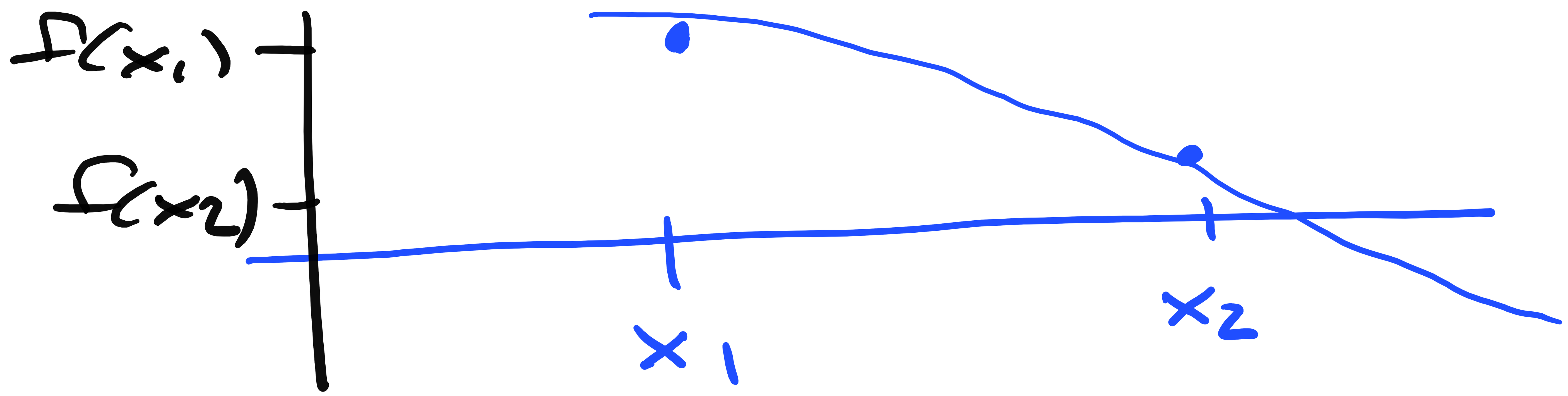
if $x_1 < x_2$ then $f(x_1) < f(x_2)$.



f is decreasing ($\downarrow$) on an interval I

if for any $x_1, x_2 \in I$,

if $x_1 < x_2$ then $f(x_1) > f(x_2)$



Thm: ⓐ $f'(x) > 0$ on interval I

$\Rightarrow f(x)$ is increasing on I.

ⓑ $f'(x) < 0$ on interval I

$\Rightarrow f(x)$ is decreasing on I.

Pf ⓐ. Assume $f'(x) > 0$ on I.

Select any $x_1 < x_2$, $x_1, x_2 \in I$.

f is diff'ble on $I \Rightarrow f$ is contin on $[x_1, x_2]$.

By the Mean Value thm,

there exists $c \in (x_1, x_2)$ such that

$$\underbrace{0 < f'(c)}_{\substack{\text{by hypothesis} \\ f'(x) > 0 \\ x \in I.}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$0 \cdot \underbrace{(x_2 - x_1)}_{+} < f(x_2) - f(x_1)$$

$$f(x_1) < f(x_2)$$

$\therefore$ f is $\uparrow$ on I.

(b) Proof of (b) is similar to proof of (a).

NOTE: If a function is CONTINUOUS on an interval I, it cannot go from $\uparrow$ to $\downarrow$ OR $\downarrow$ to $\uparrow$ at $x = c$ unless $f'(c) = 0$ OR $f'(c)$ DNE.

BEWARE: It might not change at such points.

Example $f(x) = 6x^5 + 15x^4 - 30x^3 + 7$

Find where $f \uparrow$ and where $f \downarrow$.

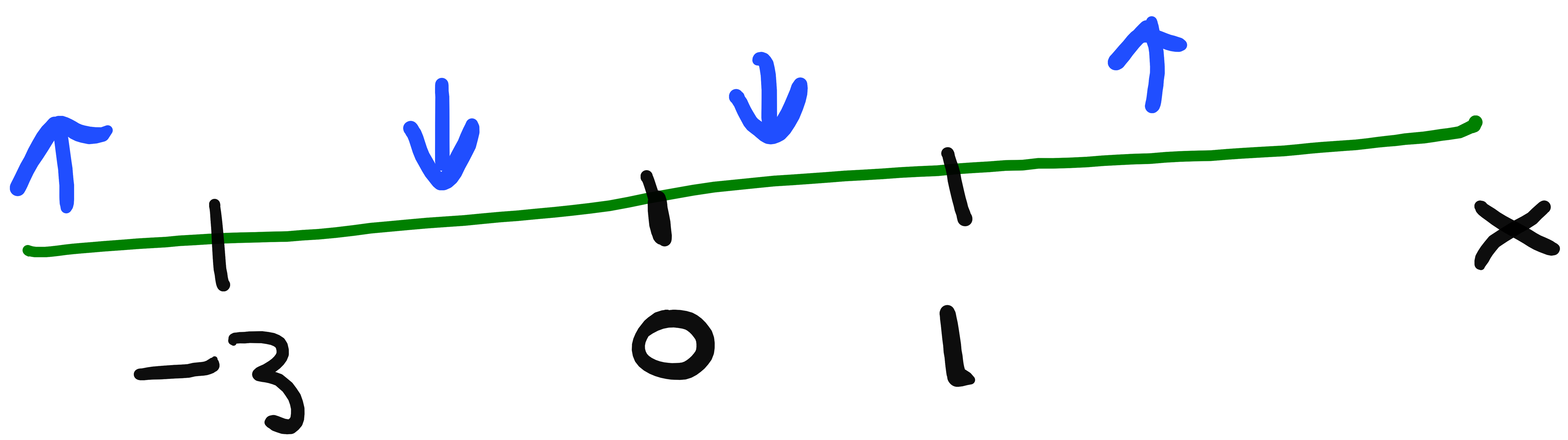
Sol'n: $f'(x) = 30x^4 + 60x^3 - 90x^2$

exists for all $x \in \mathbb{R}$.

$\therefore$ only consider where $f'(x) = 0$.

$$f'(x) = 0 \iff 30x^2(x^2 + 2x - 3) = 0$$

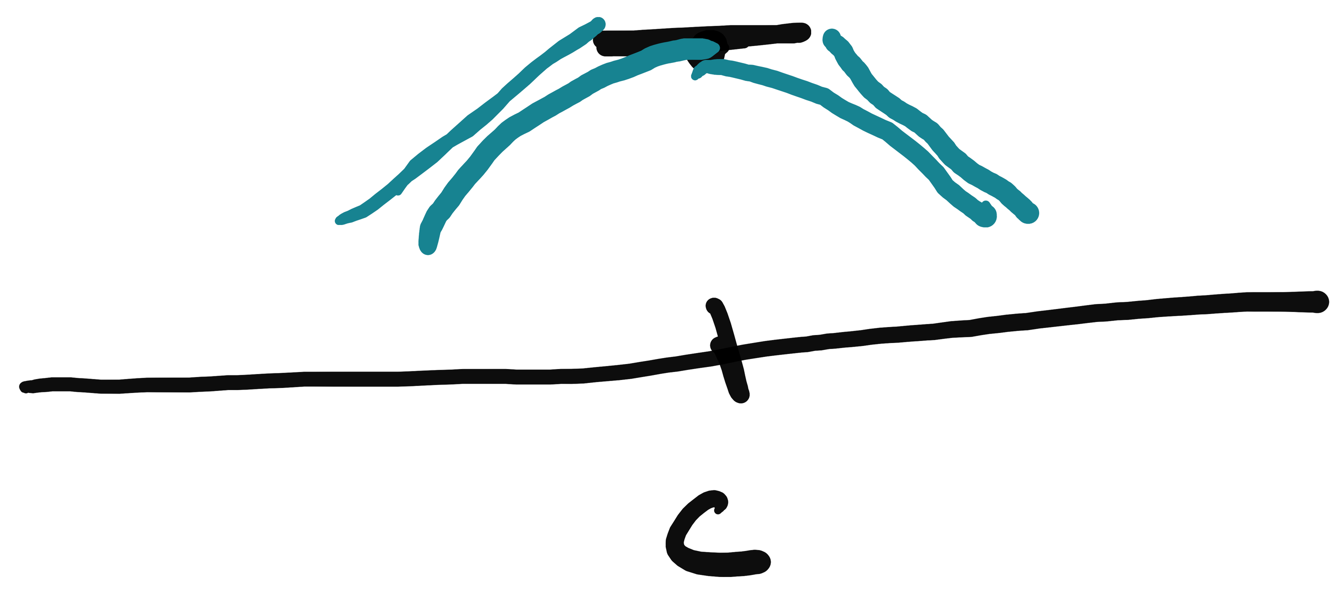
$$30x^2(x+3)(x-1) = 0.$$



Interval	$(x+3)$	x^2	$(x-1)$	$f'(x)$	$\downarrow\uparrow$
$(-\infty, -3)$	$-$	$+$	$-$	$+$	$\uparrow$
$(-3, 0)$	$+$	$+$	$-$	$-$	$\downarrow$
$(0, 1)$	$+$	$+$	$-$	$-$	$\downarrow$
$(1, \infty)$	$+$	$+$	$+$	$+$	$\uparrow$

First Derivative Test (to find local max + local mins.).

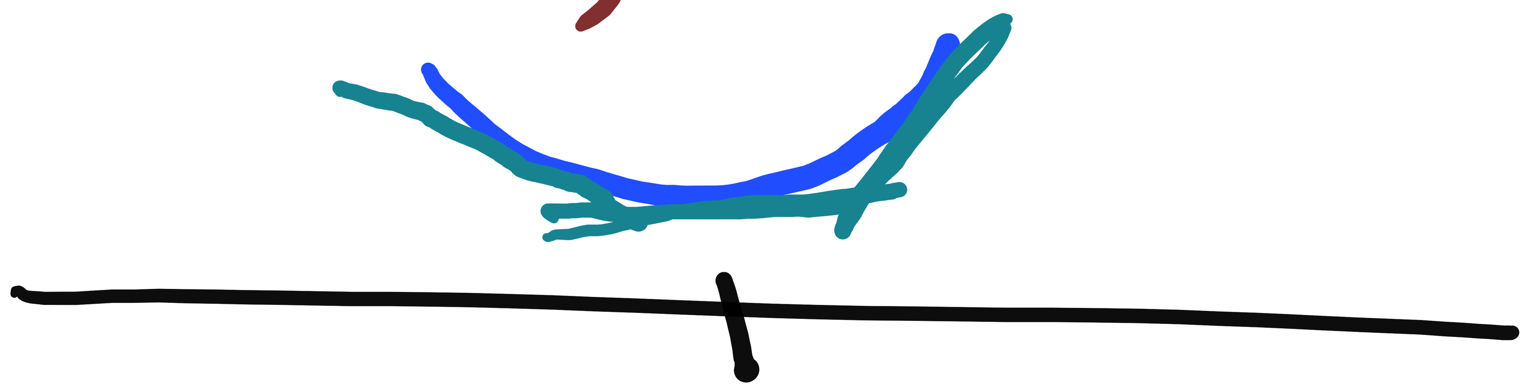
Assume $\begin{cases} f \text{ continuous and} \\ f'(c) = 0 \\ \text{or} \\ f'(c) \text{ DNE} \end{cases}$



(a) If $f'(x) > 0$ if $x < c$, near c
$f'(x) < 0$ if $x > c$, near c.

then f has a local max at $x = c$.

(b) If $f'(x) < 0$ if $x < c$, near c
$f'(x) > 0$ if $x > c$, near c

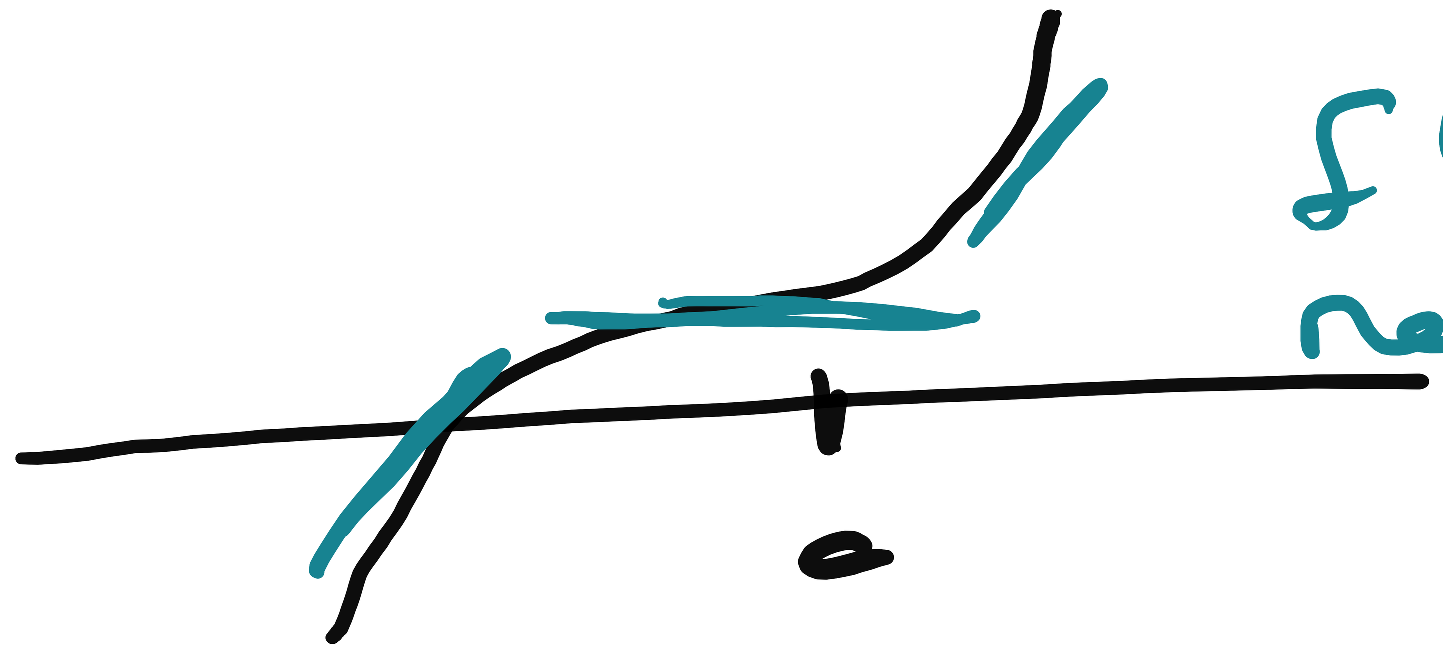


then f has a local min at $x = c$.

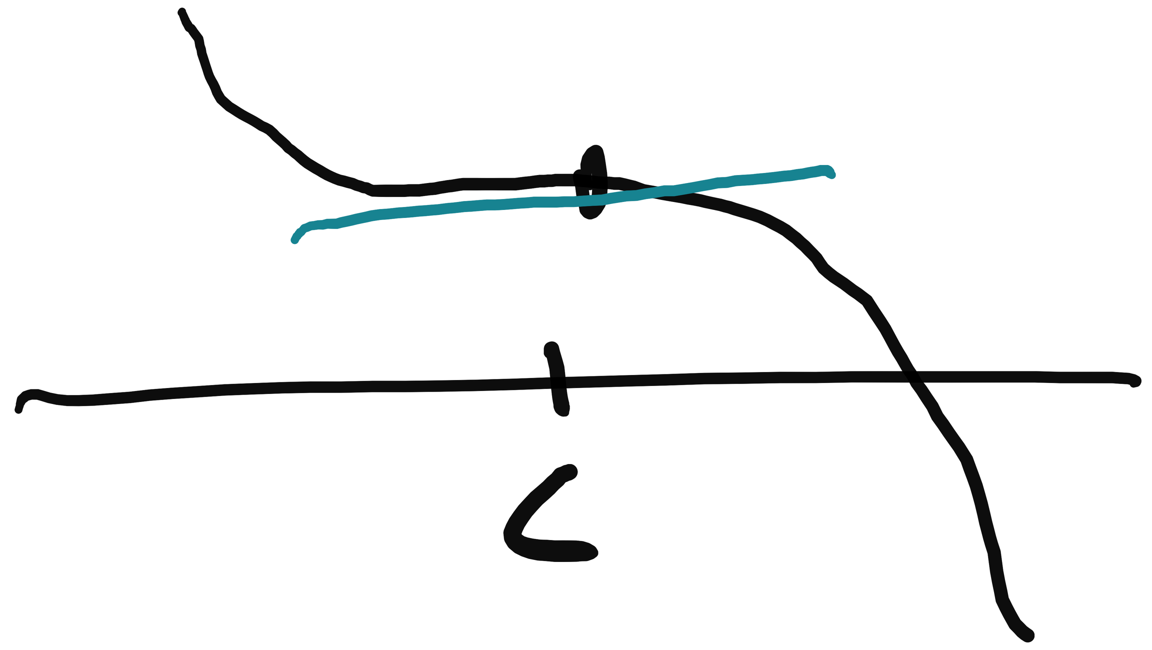
(c) If $f'(x) > 0$ for all x near c

OR If $f'(x) < 0$ for all x near c

then f does not have a local max and it does NOT have a local min at $x = c$.



$f'(x) > 0$
near c



$f'(x) < 0$
near c

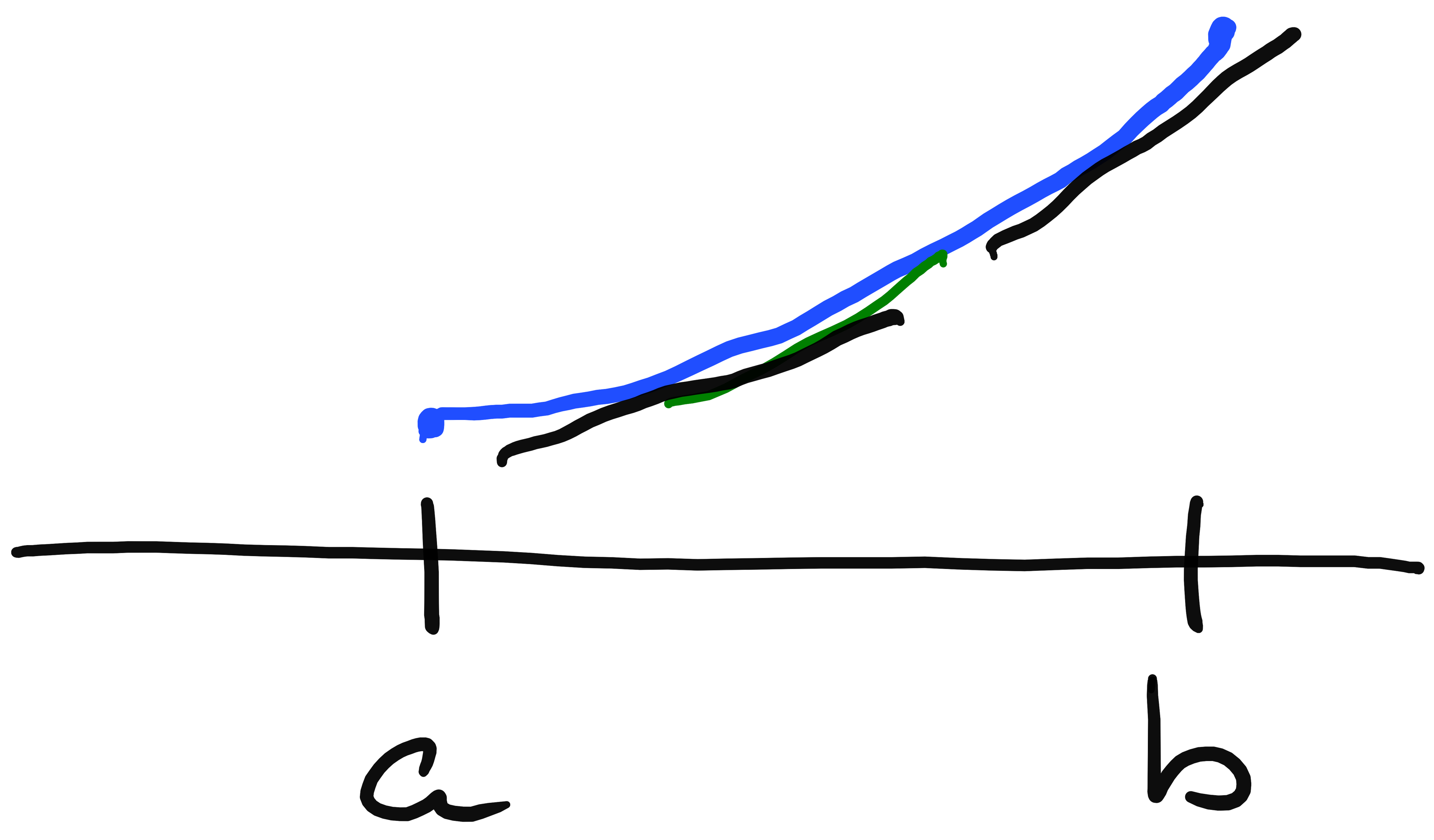
no local max
no local min.

How can the 2^{nd} derivative, $f''(x)$ help?

If $f''(x) > 0$ on interval I,
then $f'(x) \uparrow$ on I.

If $f''(x) < 0$ on interval I,
then $f'(x) \downarrow$ on I.

What else?



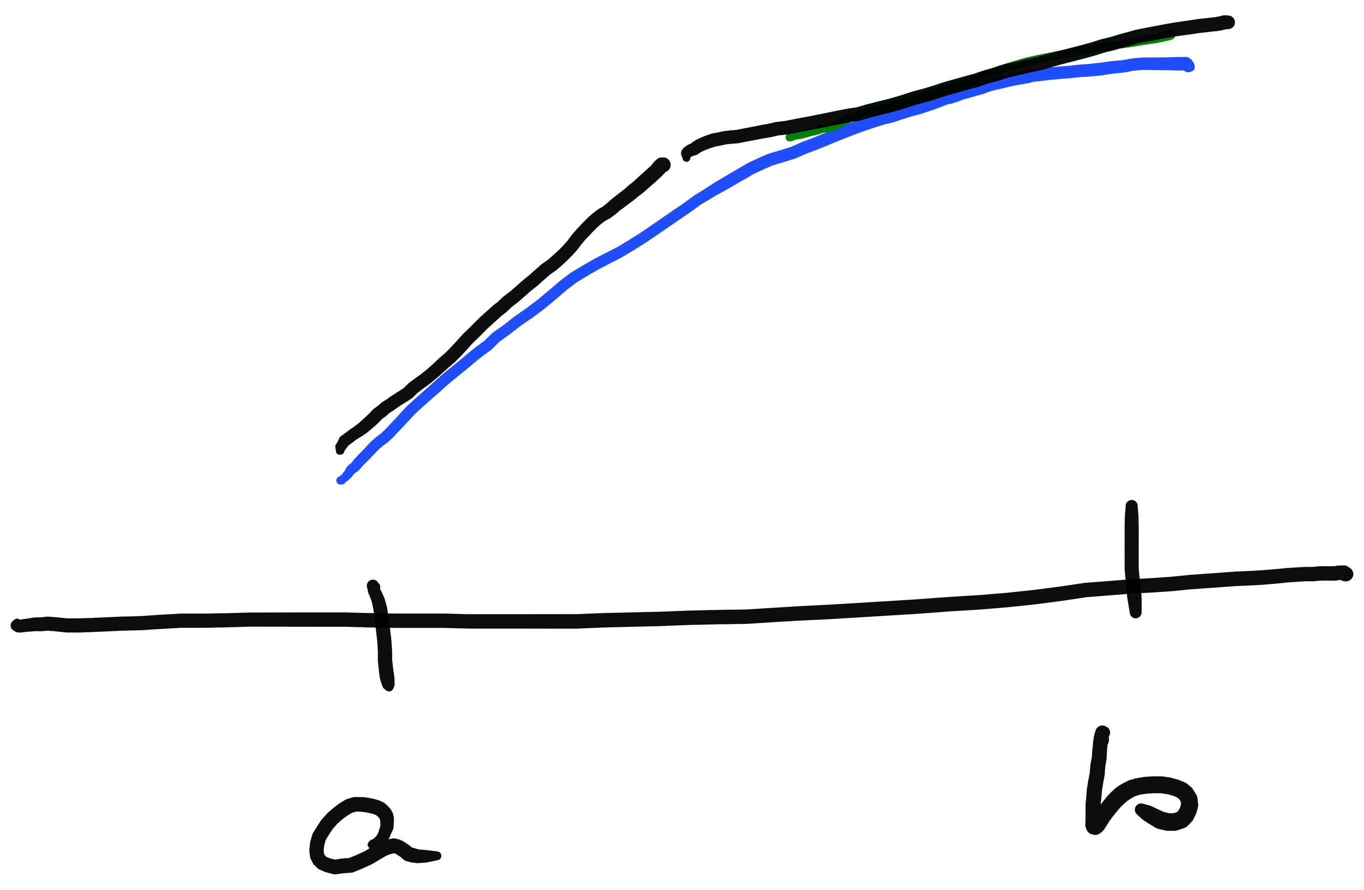
$f'(x) > 0$

$f''(x) > 0$

$\Rightarrow f'(x) \uparrow$

concave upward

- tangent to the curve at each point lies below the graph



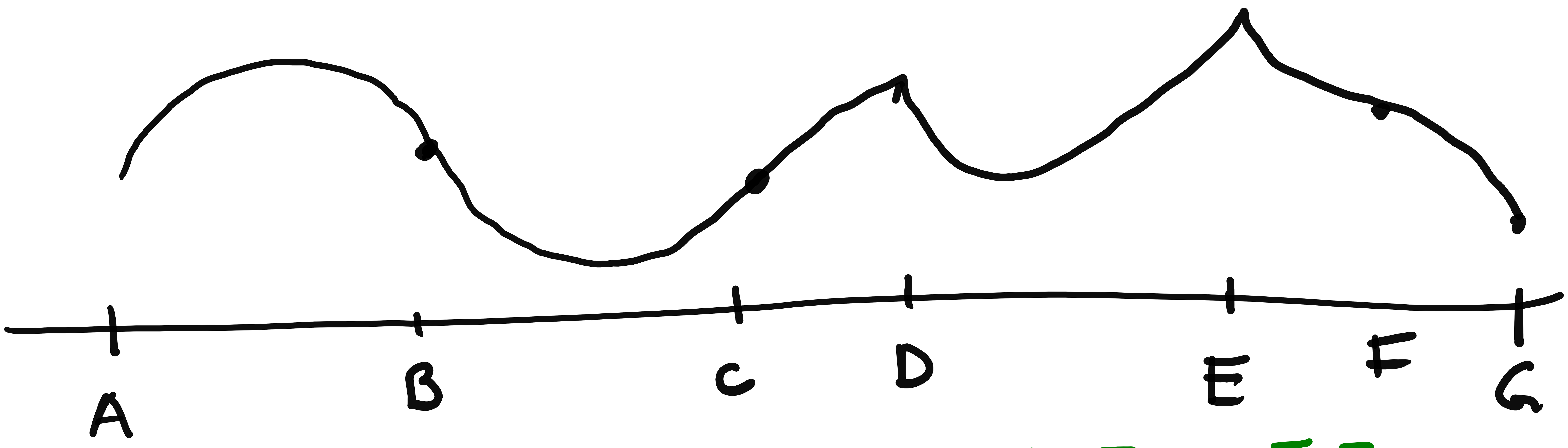
$f'(x) > 0.$

$f''(x) < 0$

$f'(x) \downarrow$

concave downward.

- tangent to the curve lies above the graph.



A–B	B–C	C–D	D–E	E–F	FG
concave down	concave up	concave down	concave up	concave up	concave down.

Def'n. f is continuous.
Inflection point of f is where f changes its concavity.

What are the inflection pts : B, C, D F
E is NOT an inflection point

Concavity Test.

Let I be an interval.

(a) If $f''(x) > 0$ for all $x \in I$, f is concave up. (think x^2)

(b) If $f''(x) < 0$ for all $x \in I$, f is concave down (think $-x^2$)

NOTE. f is continuous on I.

f can change concavity on I.

where $f''(c) = 0$ OR $f'(c)$ DNE.

OR $f''(c)$ DNE.

But $f''(c) = 0$ might NOT indicate a change in concavity.

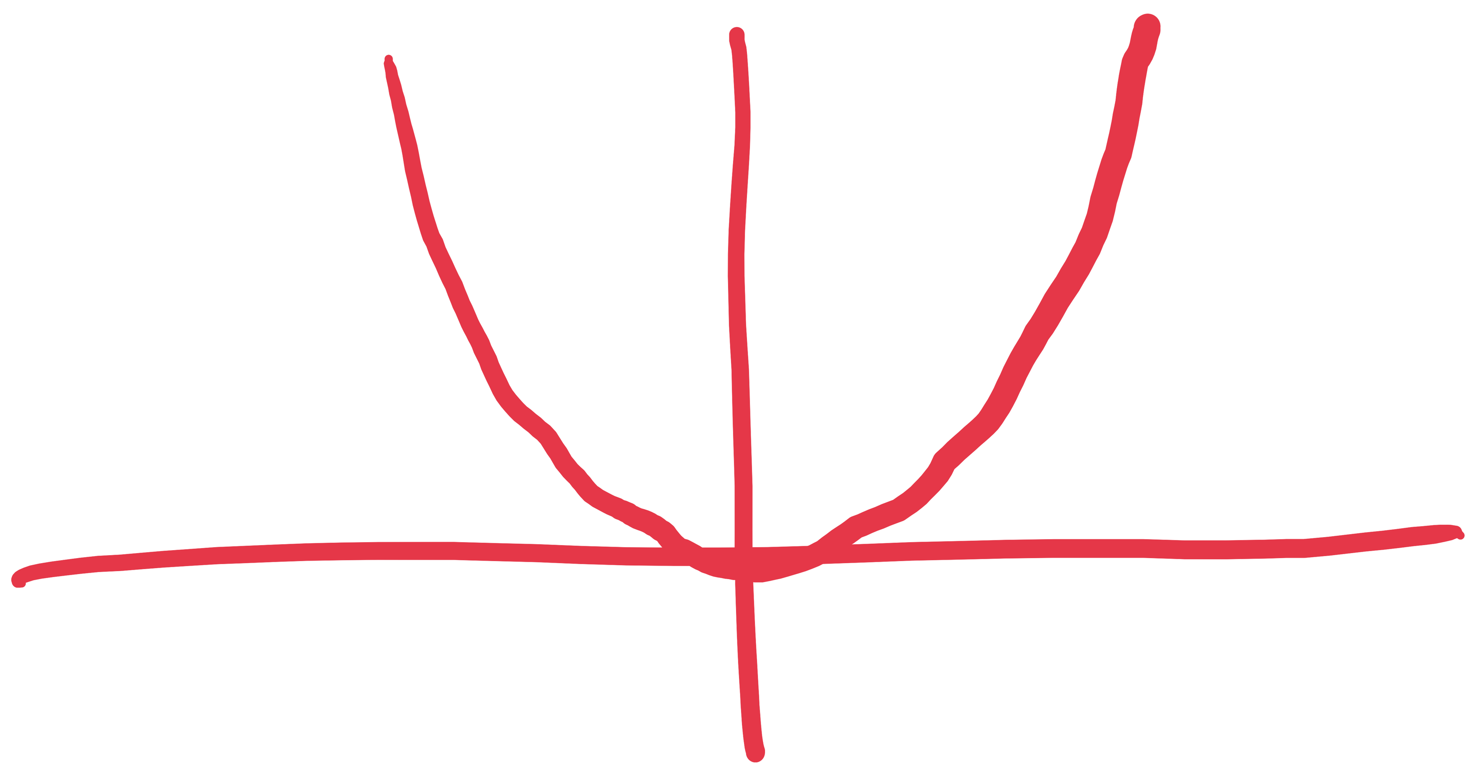
Example:

$f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

$f''(0) = 0$.



No change in concavity.

2nd Derivative Test for loc. max's and loc. min.