§4.2 Rolle's Thm & Mean Value Thm (MVTh).

Rolle's Thm

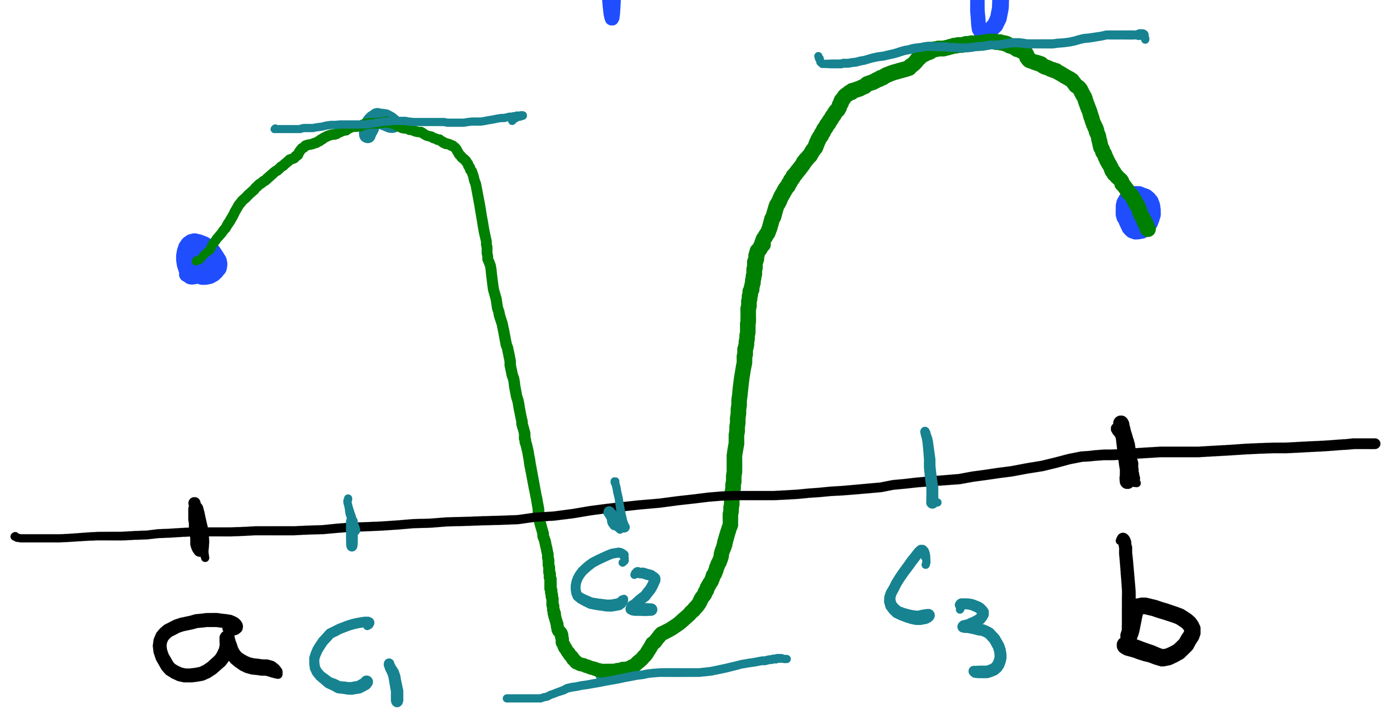
Assume 1. f is continuous on $[a,b]$

2. f is diff'ble on (a,b).

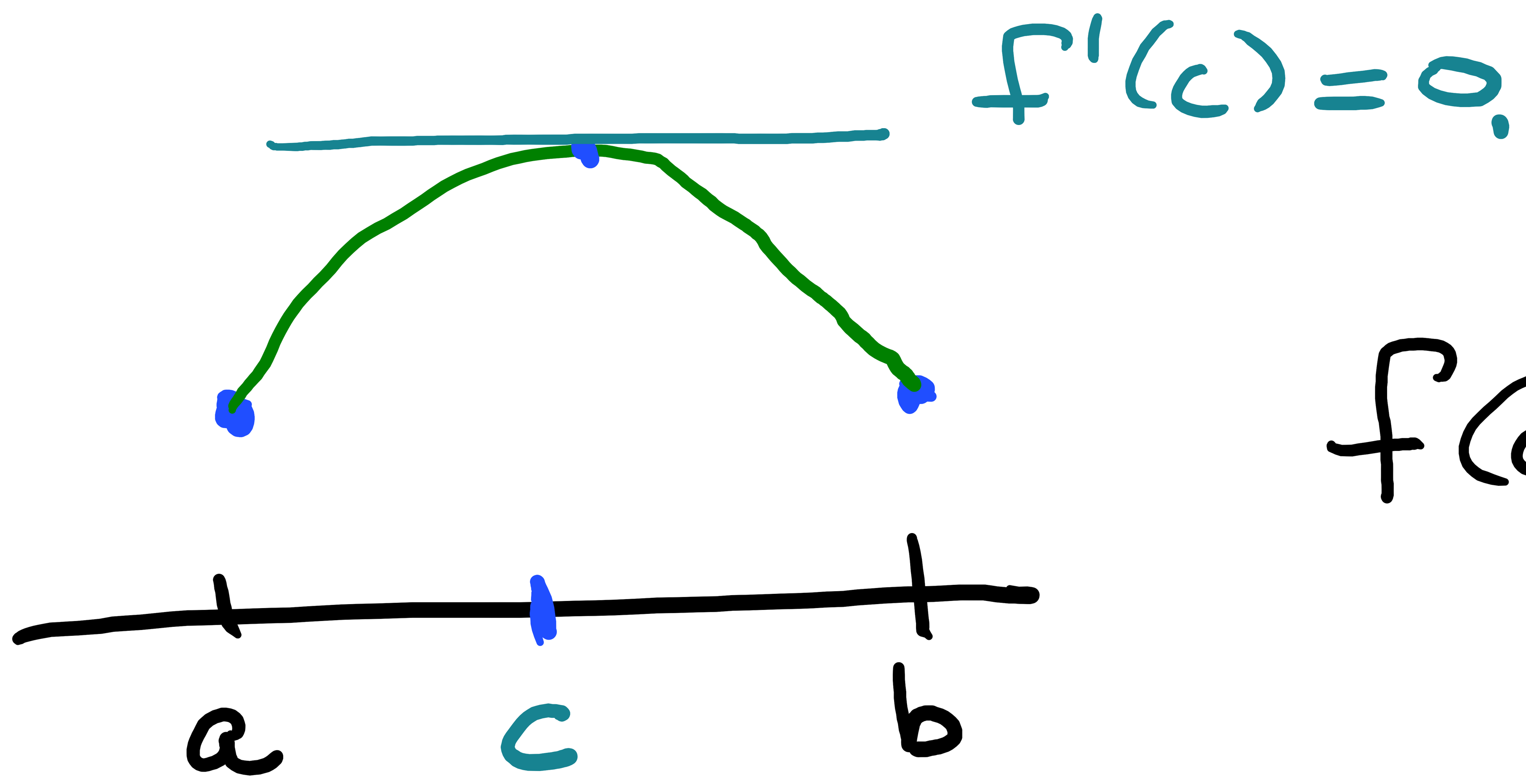
3. $f(a) = f(b)$.

Then, there is a number $c \in (a,b)$ such that $f'(c)=0$.

Picture proof.



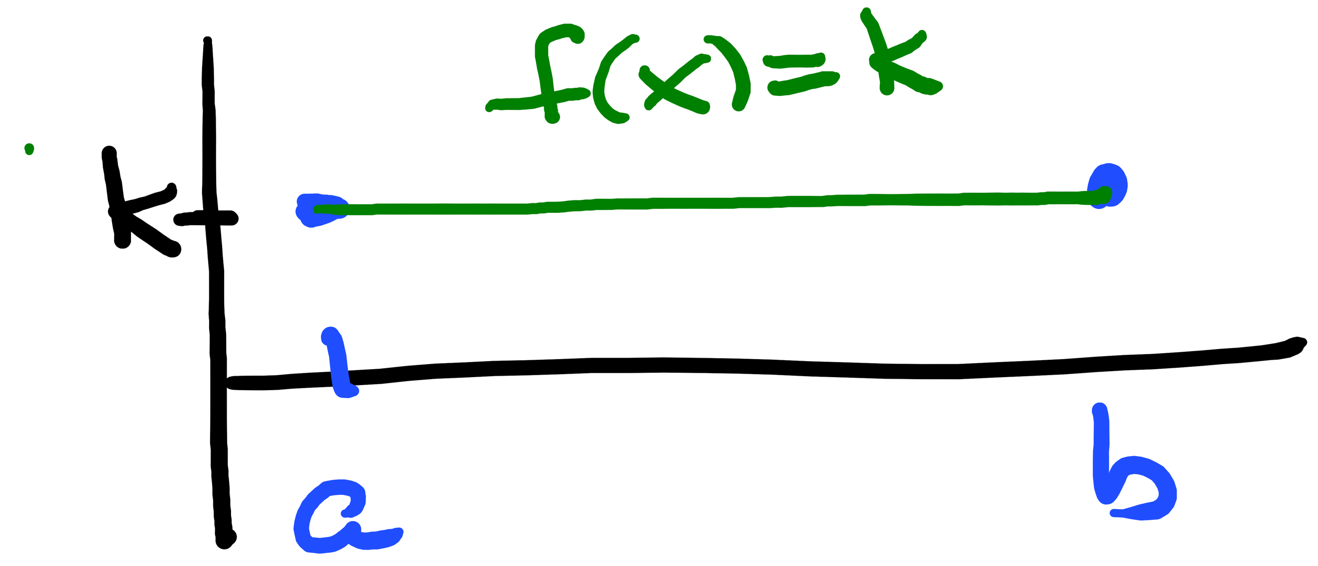
$f'(c_1)=0, f'(c_2)=0, f'(c_3)=0$



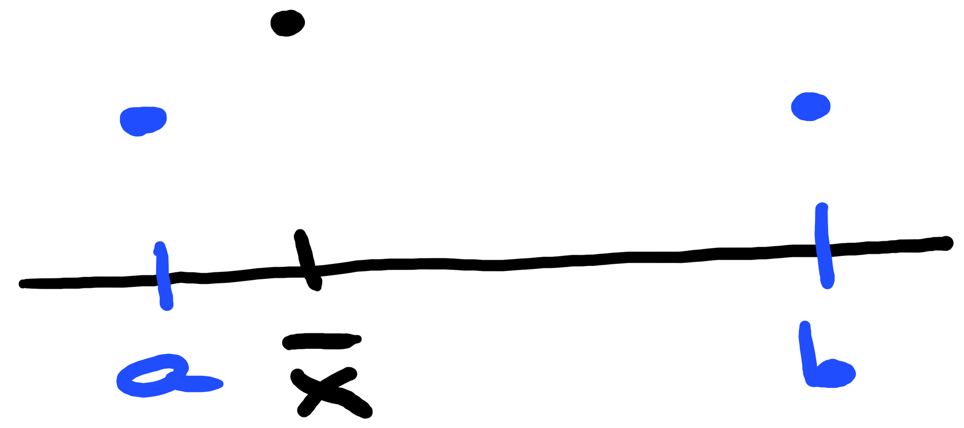
$f(a)=f(b)$

Proof. <u>Case 1.</u> $f(x)=k$, a constant for all $x \in (a, b)$

$\therefore f'(x)=0$, for all $x \in (a,b)$.



<u>Case 2.</u> Assume there is at least one $\bar{x} \in (a,b)$ with $f(\bar{x}) > f(a)$



$f(\bar{x}) > f(a)$
$f(a)=f(b)$

By the "Extreme Value Thm" f has an absolute maximum value attained at some $c \in (a,b)$, noting $f(b)=f(a)<f(\bar{x})$, so the abs. max. value is not attained at an end point.

$\therefore f$ has a "local" max. value at c.

Since f is diff'ble on (a,b) and $c \in (a,b)$

$\therefore f$ is diff'ble at c, and so $f'(c)=0$ by Fermat's Theorem.

<u>Case 3.</u> If there exists $\bar{x} \in (a,b)$ with $f(\bar{x}) < f(a)$, the proof is similar.

Mean Value Theorem (MVTh).

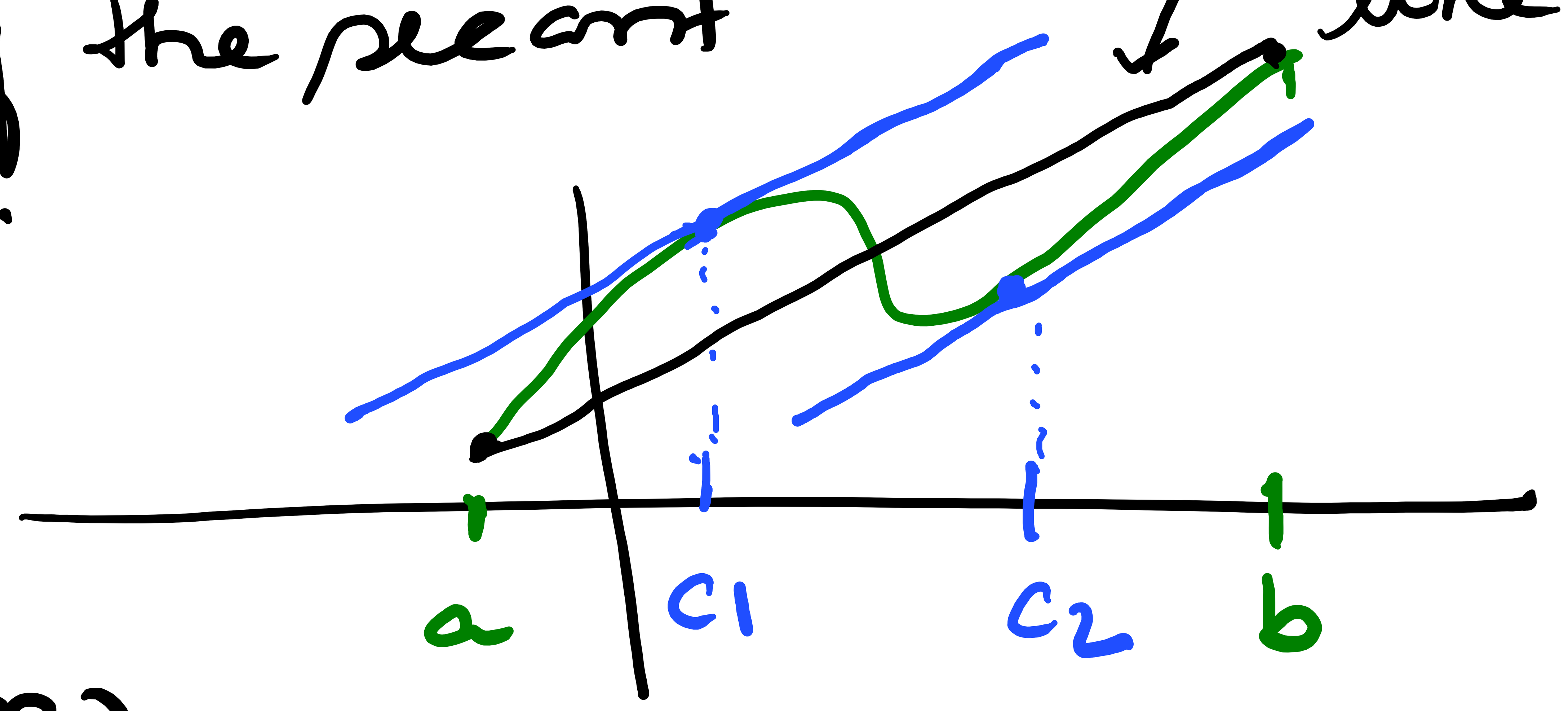
Assume that
(1) f is continuous on $[a,b]$
(2) f is diff'ble on (a,b).

Then, there is a number $c \in (a,b)$ such that

$$f'(c) = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope of the secant line!}}.$$

$\exists$ there exists

$\forall$ for all.



Proof: (Use Rolle's Thm).

Define $h(x) = f(x) - mx$, where $m = \dfrac{f(b)-f(a)}{b-a}$.

$$h(a) = f(a) - ma$$

$$h(b) = f(b) - mb.$$

To show $h(a) = h(b)$

We will show $h(b) - h(a) = 0.$

$$h(b) - h(a) = \left(f(b) - \left(\frac{\overset{m}{f(b) - f(a)}}{b-a}\right)b\right)$$

$$- \left(f(a) - \left(\frac{f(b) - f(a)}{b-a}\right)a\right)$$

$$= f(b) - f(a) - \frac{\overset{m}{(f(b) - f(a))}}{\cancel{(b-a)}} \cancel{(b-a)}$$

$$= 0.$$

$h(x) = f(x) - mx$ is contin. on $[a,b]$ $\quad m = \dfrac{f(b) - f(c)}{b-a}$ a constant.

$h'(x) = f'(x) - m$

is diff'ble on (a,b).

$h(a) = h(b)$.

By Rolle's Thm, there is a $c \in (a,b)$ such that $h'(c) = 0$.

$$0 = h'(c) = f'(c) - \left(\overbrace{\frac{f(b)-f(a)}{b-a}}^{m}\right)$$

$$\therefore \; f'(c) = \frac{f(b)-f(a)}{b-a}$$

<u>Example</u> Assume $f(x)$

1) is diff'ble for all $x \in \mathbb{R}$.
2) $f(1) = 2$
3) $f'(x) < 10$ for all $x \in \mathbb{R}$.

Find an <u>upper bound</u> for $f(4)$.

i.e. What is the largest that $f(4)$ can be?

<u>Sol'n</u>. f is diff'ble for all $x \in \mathbb{R}$.

$\Rightarrow f$ is continuous for all $x \in \mathbb{R}$.

f is continuous for $x \in [1, 4]$.

Also, f is diff'ble for $x \in (1, 4)$.

By the Mean Value Theorem, there is a $c \in (1, 4)$ such that

$$10 > f'(c) = \frac{f(4) - f(1)}{4-1} = \frac{f(4) - 2}{3}$$

since $f'(x) < 10$ for all $x \in \mathbb{R}$

$$\therefore \quad 10 > \frac{f(4)-2}{3}$$

$$30 > f(4) - 2$$

$$32 > f(4)$$

$\therefore$ $f(4)$ is less than $\underline{32}$.

Other consequences of the MVTh.

Thm/ If $f'(x) = 0$ for all x in an interval (a,b), then $f(x)$ is constant.

Proof. Take any $x_1, x_2 \in (a,b)$ and let $x_1 < x_2$.

f is cont on $[x_1, x_2]$

diff'ble on (x_1, x_2)

$\therefore$ by the MVTh.

$$\underbrace{0 = f'(c)}_{\text{since } f'(x)=0 \text{ on } (a,b)} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{for some } c \in [x_1, x_2]$$

$\Rightarrow f(x_1) = f(x_2)$.

$\therefore f(x)$ is constant on (a,b) $\cancel{}$.

Corollary

If $f'(x) = g'(x)$ for all $x \in$ on interval (a,b)

then $f(x) - g(x) = k$, where k is a constant for all $x \in (a,b)$.

Proof: Let $h(x) = f(x) - g(x)$

Then $h'(x) = f'(x) - g'(x) = 0$ by hypothesis on (a,b).

$\therefore \quad h(x) = k$, for some constant k for all $x \in (a,b)$.

$$\therefore \quad \overbrace{f(x) - g(x)}^{h(x)} = k \quad \text{for all } x \in (a,b).$$

BEWARE: $f(x) = \dfrac{x}{|x|} = \begin{cases} 1 & x > 0, \\ -1 & x < 0 \end{cases}$

Domain of f is $D = (-\infty, 0) \cup (0, \infty)$

But $f(x) \neq k$, a constant for all $x \in D$, even though $f'(x) = 0$ for all $x \in D$.

D is NOT an interval, so this does not contradict the theorem.