§3.6.

BEWARE:

p constant
x variable.

$$\begin{cases} \frac{d}{dx} x^p = p x^{p-1} & \text{(power rule)} \\ \frac{d}{dx} p^x = p^x \ln p & \text{(differentiating exponentials)} \end{cases}$$

Chain Rule.

Example: $\frac{d}{dx} \ln(g(x)) = \frac{1}{g(x)} g'(x).$

Example $f(x) = \ln|x|$. Find $f'(x)$.

Sol'n: $f(x) = \begin{cases} \ln(x) & x > 0 \\ \ln(-x) & x < 0 \end{cases}$

not defined if $x = 0$.

$$f'(x) = \begin{cases} \frac{1}{x} & x > 0. \\ \frac{-1}{(-x)} & x < 0 \end{cases}$$

$$\boxed{f'(x) = \frac{1}{x} \quad x \neq 0.}$$ Important fact.

Logarithmic Differentiation

$$y = \frac{e^{2x}(x^2-1)^5}{\sqrt{x^3 - x^2 + 2}}.$$ Find y'.

Sol'n: Could use product rule & quotient rule.
(HW try it).

Instead, take ln of both sides and differentiate.

$$\ln|y| = \ln\left(\frac{e^{2x}(x^2-1)^5}{\sqrt{x^3-x^2+2}}\right)$$

$$\ln|y| = \ln e^{2x} + \ln(x^2-1)^5 - \ln(\sqrt{x^3-x^2+2})$$
$$= 2x + 5\ln(x^2-1) - \frac{1}{2}\ln(x^3-x^2+2)$$

Diff. Implicitly wrt x

$$\frac{1}{y}y' = 2 + \frac{5}{x^2-1}(2x) - \frac{1}{2}\,\frac{3x^2-2x}{x^3-x^2+2}$$

Solved for y':

$$y' = \underbrace{\frac{e^{2x}(x^2-1)^5}{\sqrt{x^3-x^2+2}}}_{y}\left\{2 + \frac{10x}{x^2-1} - \frac{3x^2-2x}{2(x^3-x^2+2)}\right\}$$

Example $\frac{d}{dx}(g(x))^{h(x)}$

Sol'n.

$$y = (g(x))^{h(x)}$$

$$\ln y = h(x)\ln(g(x))$$

$$\frac{1}{y}y' = h'(x)\ln(g(x)) + h(x)\frac{d}{dx}\ln(g(x))$$

$$\frac{1}{y}y' = h'(x)\ln g(x) + h(x)\frac{g'(x)}{g(x)}$$

$$y' = \underbrace{(g(x))^{h(x)}}_{y}\left\{h'(x)\ln g(x) + h(x)\frac{g'(x)}{g(x)}\right\}$$

Example: $y = x^{\tan x}$. Find $\frac{dy}{dx}$.

Sol'n: $\ln y = \tan x \ln x$

Diff. implic.

$$\frac{1}{y} y' = \sec^2 x \ln x + \frac{\tan x}{x}$$

$$y' = \underbrace{x^{\tan x}}_{y} \left\{ \sec^2 x \ln x + \frac{\tan x}{x} \right\}.$$

§3.11. Hyperbolic Functions.

hyperbolic sine $\quad \sinh(x) = \frac{e^x - e^{-x}}{2}$

hyperbolic cosine. $\quad \cosh(x) = \frac{e^x + e^{-x}}{2}$.

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

Properties. ① $\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$

$$\sinh(-x) = \frac{e^{-x} - e^{x}}{2} = -\sinh(x)$$

$\therefore$ $\sinh(x)$ is an ODD function.

(2) $$\cosh(x) = \frac{e^{x}+e^{-x}}{2}$$

$$\cosh(-x) = \frac{e^{-x}+e^{x}}{2} = \cosh(x)$$

$\therefore$ $\cosh(x)$ is an EVEN function.

Hyperbolic Identities.

$$\boxed{\cosh^2(x) - \sinh^2(x) = 1}$$

Pf: $$\cosh^2(x) - \sinh^2(x)$$

$$= \left(\frac{e^{x}+e^{-x}}{2}\right)^2 - \left(\frac{e^{x}-e^{-x}}{2}\right)^2$$

$$= \left(\frac{\cancel{e^{2x}}+2+\cancel{e^{-2x}}}{4}\right) - \left(\frac{\cancel{e^{2x}}-2+\cancel{e^{-2x}}}{4}\right)$$

$$= \frac{2}{4} + \frac{2}{4} = 1$$

$$\boxed{1 - \tanh^2(x) = \operatorname{sech}^2(x)}$$

Pf: $\cosh^2(x) - \sinh^2(x) = 1$

divide both sides by $\cosh^2(x)$.

$$\frac{\cosh^2(x)}{\cosh^2(x)} - \frac{\sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}$$

$$1 - \tanh^2(x) = \operatorname{sech}^2(x) \quad ✓$$

$$\boxed{\sinh(2x) = 2\sinh(x)\cosh(x)}$$

Pf. $\sinh(2x) = \dfrac{e^{2x} - e^{-2x}}{2}$

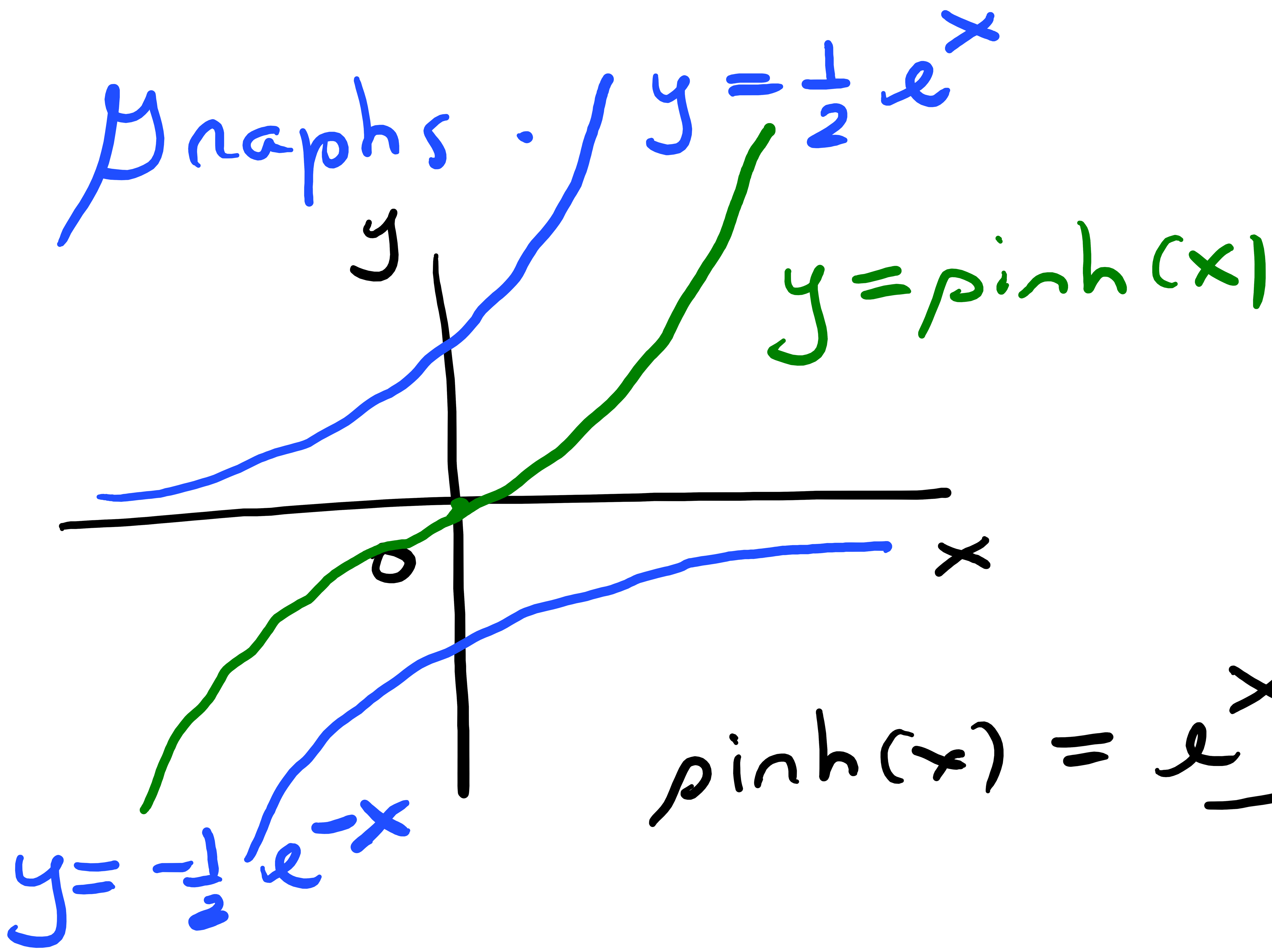
$$= \frac{1}{2}\left((e^{x})^{2} - (e^{-x})^{2} \right)$$

$$= \frac{1}{2}\left(e^{x}-e^{-x}\right)\left(e^{x}+e^{-x}\right)$$

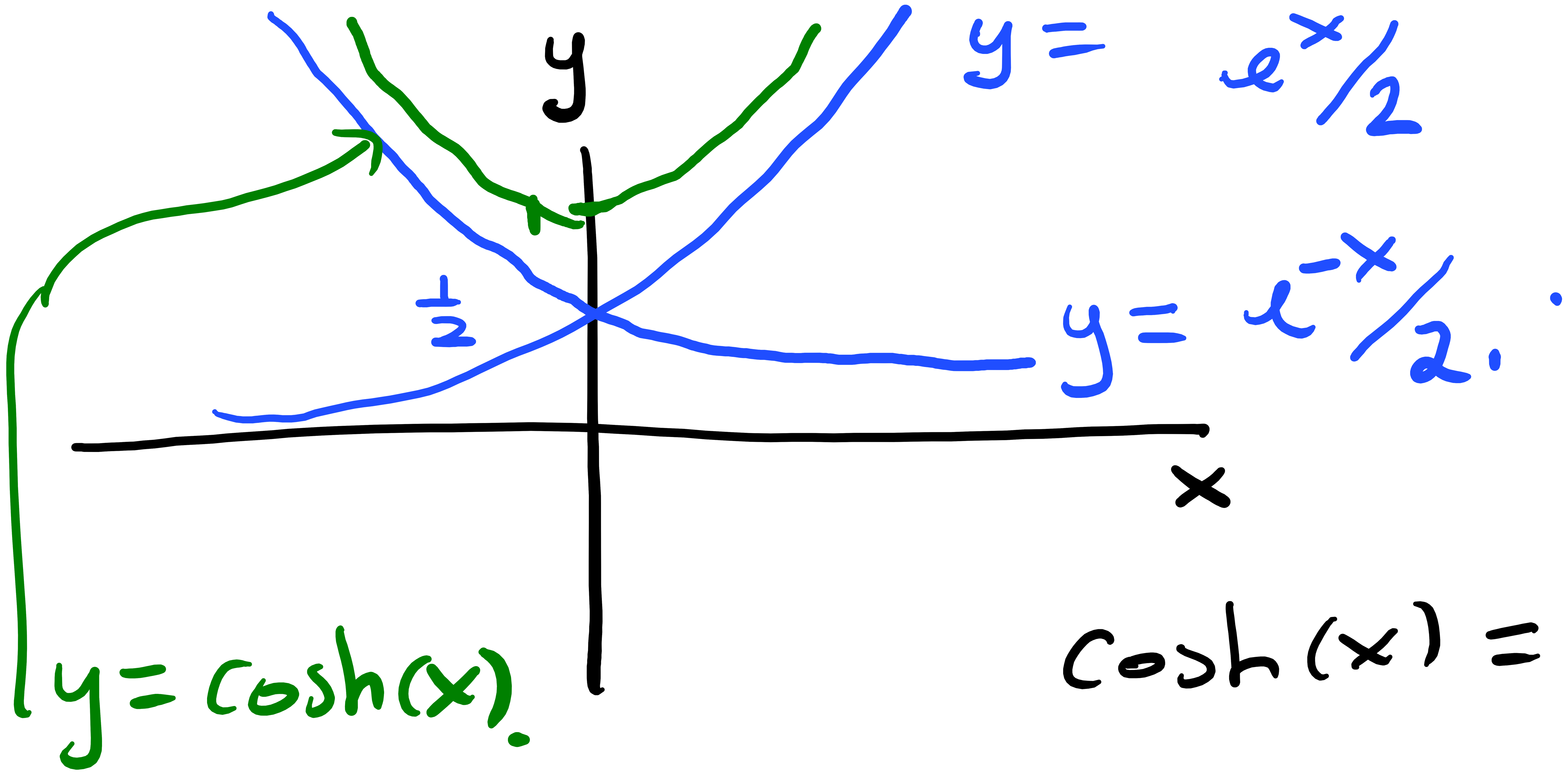
$$= \frac{(2)\cancel{(2)}}{\cancel{2}} \frac{\left(e^{x}-e^{-x}\right)}{2} \frac{\left(e^{x}+e^{-x}\right)}{2}$$

$$= 2\sinh(x)\cosh(x).$$

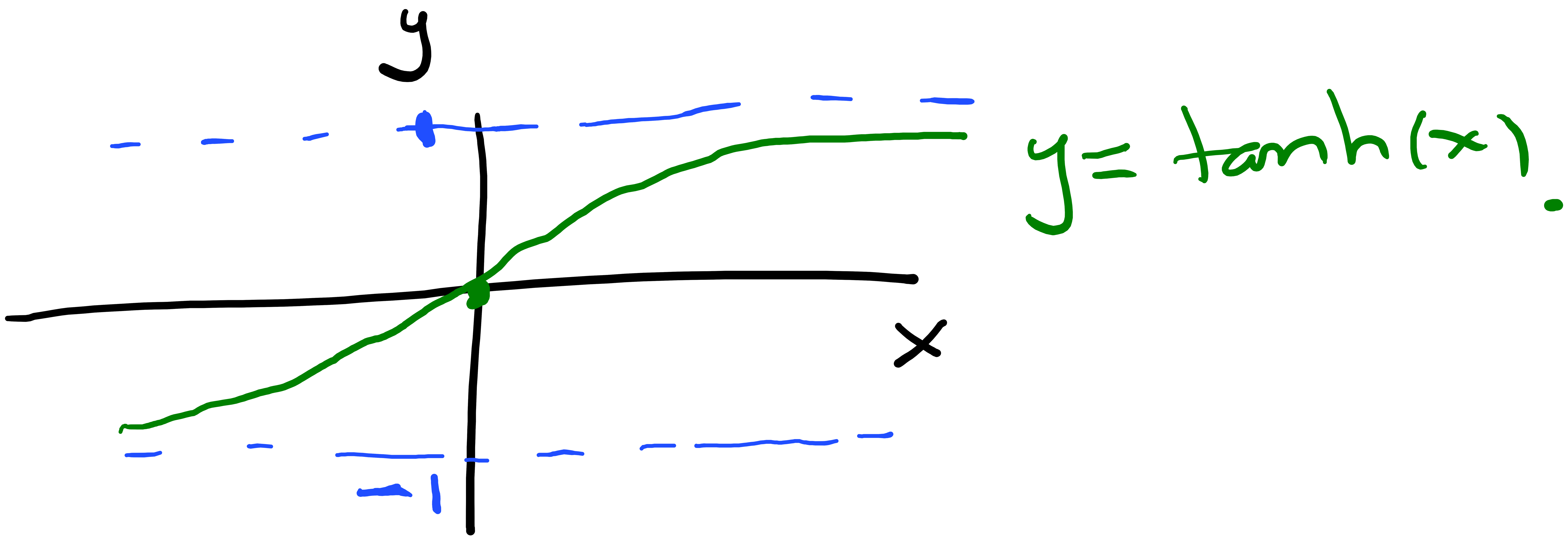
Graphs.



$$\sinh(x) = \frac{e^{x}-e^{-x}}{2}$$



$$\cosh(x) = \frac{e^x + e^{-x}}{2}.$$



$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\lim_{x\to\infty} \tanh(x) = 1$$

$$\lim_{x \to -\infty} \tanh(x) = -1$$

$$\tanh(0) = 0$$

Note: $e^x = \sinh(x) + \cosh(x)$

Derivatives.

$$\frac{d}{dx} \sinh(x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$\frac{d}{dx}\cosh(x) = \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= \frac{e^x - e^{-x}}{2}$$

$$= \sinh(x)$$

(<u>NOTE</u>: <u>no minus sign</u>)

$$\frac{d}{dx}\tanh(x) = \frac{d}{dx}\frac{(e^x - e^{-x})}{(e^x+e^{-x})}$$

$$= \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$

$$= \frac{(e^x - e^{-x})^2}{(e^x - e^{-x})^2} - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2$$

$$= 1 - \tanh^2(x)$$ (Hyperbolic trig identity)

$$= \operatorname{sech}^2(x)$$

$\therefore$

$$\boxed{\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x)}$$