§3.4 cont'd Chain Rule.

I $(f(g(x)))' = f'(g(x))g'(x)$

or.

II $(f(g(x)))'$ Leibnitz $g(x) = u$

$$\frac{df(g(x))}{dx} = \frac{df(u)}{dx} = \frac{df(u)}{du}\frac{du}{dx}.$$

Example $\frac{d}{dx}(g(x))^n = n\,g(x)^{n-1}g'(x)$

power rule & chain rule.

$$\frac{d}{dx}(\underbrace{x^2+\tan x}_{g(x)})^{35} = 35(\underbrace{x^2+\tan x})^{34}\frac{d}{dx}(x^2+\tan x)$$

$f(x) = x^{35}.$

$$= 35(x^2+\tan x)^{34}(\underbrace{2x+\sec^2 x}_{g'(x)}).$$

Example $\frac{d}{dx} \sin(x) e^{x^2}$ (product rule and chain rule)

$$= \cos(x) e^{x^2} + (\sin x) e^{x^2} \underbrace{(2x)}_{g'(x)}$$

Differentiating Exponential Functions.

We already showed $\frac{d}{dx}(e^x) = e^x$ from the definition of the derivative

Find $\frac{d}{dx} b^x$, $b>0$.

Write $b^x = (e^{\ln b})^x = e^{x \ln b}$ (since $e^{\ln b} = b$ using $f(f^{-1}(b)) = b$)

$$\frac{d}{dx} b^x = \frac{d}{dx} e^{x\ln b} = e^{x \ln b} \frac{d}{dx}(x \ln b)$$

$$= e^{x \ln b} \ln b.$$

$$= b^x \ln b \text{ using}$$

$\therefore$ $$\boxed{\begin{array}{ll} \frac{d}{dx} b^x = b^x \ln b. & \\ \frac{d}{dx} e^x = e^x & (\ln e = 1) \end{array}}$$

§3.5 Implicit Differentiation.

If an equation defines y "implicitly" as a function of x, like

$$y^4 + x^3 y = 9.$$

NOTE: We cannot solve for y explicitly as a function of x.

Then, differentiate both sides wrt x thinking of y as $y(x)$ (a function of x).

Example: Find $y'(x)$ if $y^4 + x^3 y = 9$.

Sol'n: $\frac{d}{dx}\left(y^4 + x^3 y\right) = \frac{d}{dx} 9.$

$$4y^3 \frac{dy}{dx} + 3x^2 y + x^3 \frac{dy}{dx} = 0.$$

Solve for $\frac{dy}{dx}$:

(*) $\frac{dy}{dx} = \frac{-3x^2 y}{4y^3 + x^3}$ (OK to leave this with function y).

Question: Find the equation of the tangent line to the curve $y(x)$ that satisfies. $y^4 + x^3 y = 9$ at the point $(x,y) = (2,1)$ (i.e. $y(2) = 1$).

Slope of $y(x)$ at $(2,1)$ is

$$\frac{dy}{dx} = \frac{-3x^2y}{4y^3+x^3}$$ from ⊛

$x=2$
$y=1.$

$$= \frac{-3(2)^2(1)}{4(1)^3+(2)^3}$$

$$= \frac{-12}{12} = -1.$$

i.e. slope is -1.

$\therefore$ The tangent line at $(2,1)$ has slope -1.
and so the equation of the tangent line is:

$$\underbrace{\frac{y-1}{x-2} = -1}_{} \Rightarrow (y-1) = -(x-2)$$

OR

$$x+y = 3.$$

using $\frac{y-y_0}{x-x_0} = m$, $y_0=1$
$x_0=2$, $m=-1$

or using $y = mx+b$.
where $m=-1$
Find b using $y=1$
$x=2$.
$1 = (-1)(2)+b \Rightarrow b=3$,
$\therefore y = -x+3$.

Derivative of an INVERSE FUNCTION.

Show. $\boxed{\dfrac{d}{dx} f^{-1}(x) = \dfrac{1}{f'(f^{-1}(x))}}$ know.

Sol'n

Let $y = f^{-1}(x)$ (*) . To find $\dfrac{dy}{dx}$.

$$f(y) = f(f^{-1}(x))$$

$$\therefore f(y) = x$$

Differentiate both sides implicitly thinking of y as a function of x, i.e. $y = y(x)$.

$$\frac{df(y)}{dx} \longrightarrow f'(y)\frac{dy}{dx} = \frac{d}{dx} x = 1.$$

$$\therefore \frac{dy}{dx} = \frac{1}{f'(y)} \quad (\text{where } y = f^{-1}(x))$$

$$\frac{df^{-1}(x)}{dx} = \frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))} \Rightarrow \boxed{\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))}}$$

Derivatives of INVERSE TRIG FUNCTIONS.

Show $\boxed{\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}}$ know.

Sol'n: Can let $y = \sin^{-1}(x)$

$\sin(y) = x$

diff both sides implicitly

$\cos(y)\dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos(y)}$

$= \dfrac{1}{\cos(\sin^{-1}(x))}$

OR use the formula.

derived above where $f(x) = \sin x$

$\therefore \dfrac{d}{dx}\sin^{-1}(x) = \dfrac{1}{\cos(\sin^{-1}(x))}$

Now, if $\theta = \sin^{-1}(x)$

$\sin\theta = x$

$\cos(\theta) = \dfrac{\sqrt{1-x^2}}{1}$

$\dfrac{dy}{dx} = \dfrac{1}{\cos(\sin^{-1}(x))}$

$= \dfrac{1}{\cos(\theta)}$

$= \dfrac{1}{\sqrt{1-x^2}}$

Show $\boxed{\dfrac{d}{dx}\tan^{-1}(x) = \dfrac{1}{1+x^2}}$ know

Sol'n: $\dfrac{d}{dx}\tan^{-1}(x) = \dfrac{1}{\sec^2(\tan^{-1}(x))}$

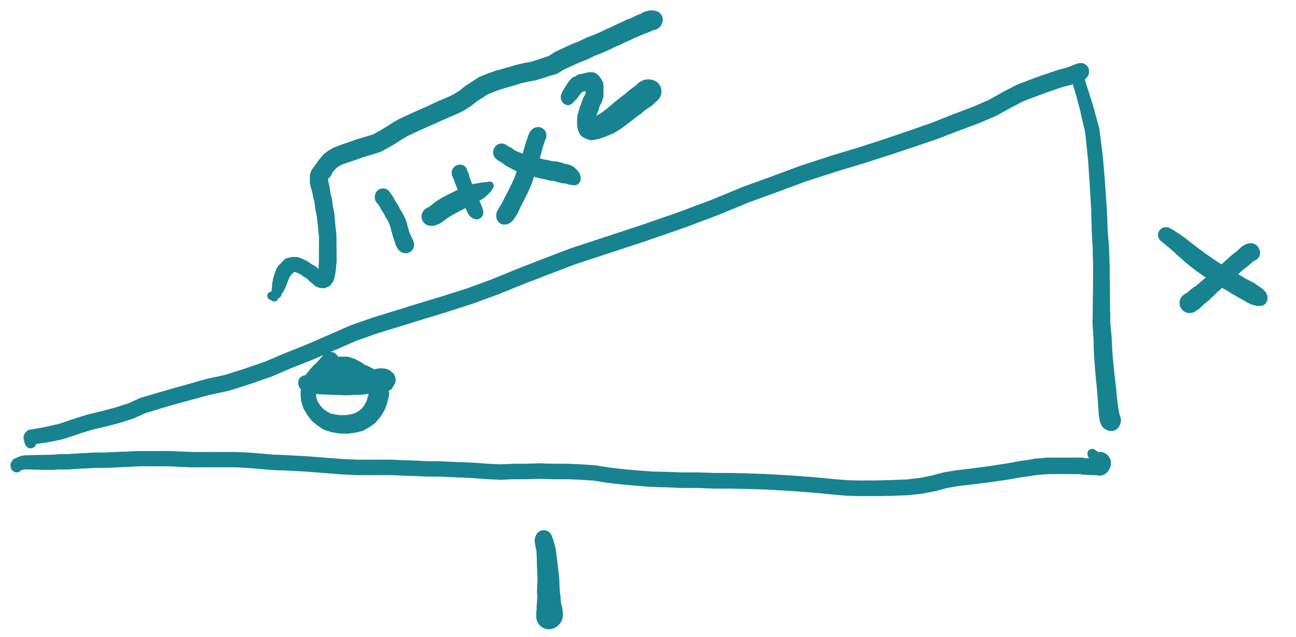
using formula where $f(x)=\tan x$

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

$$= \cos^2(\underbrace{\tan^{-1}(x)}_{\theta})$$

Let $\theta = \tan^{-1}(x)$

$\tan\theta = x$



$$\cos(\theta) = \frac{1}{\sqrt{1+x^2}}$$

$$\cos(\underbrace{\tan^{-1}(x)}_{\theta}) = \frac{1}{\sqrt{1+x^2}}.$$

$\cos^2(\theta)$

$$\Rightarrow \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

HW. Show $\boxed{\dfrac{d}{dx}\cos^{-1}(x) = \dfrac{-1}{\sqrt{1-x^2}}}$ know.

Be able to derive $\dfrac{d}{dx}\csc^{-1}(x)$, $\dfrac{d}{dx}\sec^{-1}(x)$, and $\dfrac{d}{dx}\cot^{-1}(x)$.

§ 3.6. Derivatives of Logarithmic Functions.

Show $\frac{d}{dx} \log_b x = \frac{1}{x \ln(b)}$

& hence $\frac{d}{dx} \ln x = \frac{1}{x}$

know these.

Sol'n. RECALL that $\frac{d}{dx} b^x = b^x \ln b$

Let $y = \log_b x$ (Want $\frac{dy}{dx} = \frac{d}{dx} \log_b x$).

Then $b^y = b^{\log_b x}$, but $b^{\log_b x} = x$ since $f(f^{-1}(x)) = x$.

$\therefore \boxed{b^y = x}$

Differentiate implicitly wrt x where $y = y(x)$.

$$\frac{d}{dx} b^y = \frac{d}{dx} x$$

$$b^y \ln b \frac{dy}{dx} = 1$$

(chain rule)

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{b^y \ln b}, \quad \text{but } b^y = x$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{1}{x \ln b}.}$$

If $b = e$, since $\ln e = 1$

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$