

Problem Solver's Toolbox

Polya's 4 stages when you are stuck:

- Understand the problem
- Make a plan
- Carry out the plan
- Look back

(with possible returns to previous stages)

L.C. Larson's Heuristic Tips

1. Search for a Pattern
2. Draw a Figure
3. Formulate an Equivalent Problem
4. Modify the Problem
5. Choose Effective Notation
6. Exploit Symmetry
7. Divide into Cases
8. Work Backwards
9. Argue by Contradiction
10. Pursue Parity
11. Consider Extreme Cases
12. Generalize

Three basic tricks in mathematics:

- Adding zero: we will see many examples of that. Zero is often disguised as $A - A$
 - Multiplication by 1: 1 is often disguised as $\frac{A}{A}$.
Of course, A may look like a complicated expression.
 - Reduction to a previously solved problem: this will accompany us through the course.
 - Carl Jacobi's imperative: Invert, always invert!
- (OK, that's four tricks.)

Induction

"Short memory" induction: A statement $P(n)$ will be proved for all integers $n \geq 0$ if

- (1) it is proved for $n = 0$
- (2) for every $n \geq 1$, $P(n - 1)$ is proved to imply $P(n)$

"Long memory" induction: A statement $P(n)$ will be proved for all integers $n \geq 0$ if

- (1) it is proved for $n = 0$
- (2) assuming that $P(m)$ is true for every $m < n$, we can prove $P(n)$.

(Remark: Formally, here part (1) can be skipped. Find out why.)

Of course, induction can be modified to start from $n = 1$, or any starting value (with the conclusion properly modified).

Bernoulli Inequality:

Suppose $n \geq 2$ is integer, and $x \neq 0$, $x > -1$. Then

$$(1+x)^n > 1+nx$$

(No calculus needed, just induction.)

Exercise. Let α be a real number, then for $x > -1$, show that

$$\frac{1+\alpha x}{(1+x)^\alpha} = 1 - \alpha(\alpha-1)x^2 \int_0^1 \frac{t}{(1+xt)^{\alpha+1}} dt$$

Now you can extend Bernoulli Inequality to non-integer real α

(distinguishing the cases $\alpha < 0$, $\alpha = 0$, $0 < \alpha < 1$, $\alpha = 1$, $\alpha > 1$).

AGM Inequality (arithmetic-geometric mean inequality):

Suppose a_1, \dots, a_n are positive numbers, not all equal. Then

$$\frac{a_1 + \dots + a_n}{n} > \sqrt[n]{a_1 \dots a_n}$$

or in sum-product notation,

$$\frac{1}{n} \sum_{j=1}^n a_j > \sqrt[n]{\prod_{j=1}^n a_j}$$

Cauchy-Schwarz Inequality

(Abstract form) In an inner product space, if vectors x and y are linearly independent then

$$|\langle x, y \rangle| < \|x\| \|y\|$$

Standard finite-dimensional case:

Suppose x_j, y_j are real scalars, for $j = 1, \dots, n$, then

$$\left| \sum_{j=1}^n x_j y_j \right| \leq \sqrt{\sum_{j=1}^n x_j^2} \sqrt{\sum_{j=1}^n y_j^2}$$
