ALGEBRAS WITH FEW SUBPOWERS ARE TRACTABLE

joint work with J. Berman, P. Idziak, R. McKenzie, M. Valeriote and R. Willard

1 GMM operation

1.1 Definition

We call a k-ary operation g on the set A a *generalized majority-minority* operation (GMM for short) when for all $\{a, b\} \subseteq A$, g on the entries from $\{a, b\}$ satisfies either the near-unanimity equations

$$\begin{array}{rcl} g(y, x, x, \ldots, x) & = & x \\ g(x, y, x, \ldots, x) & = & x \\ \vdots & & \vdots \\ g(x, x, \ldots, x, y) & = & x \end{array}$$

or the Mal'cev-like equations

$$\begin{array}{rcl} g(y, x, x, \ldots, x) & = & y \\ g(x, x, \ldots, x, y) & = & y. \end{array}$$

1.2 Generation

Let a pair of elements $f, g \in A^n$ be such that for some $1 \leq i \leq n$ $f(i) \neq g(i)$, $\{f(i), g(i)\}$ is a minority pair and for all $1 \leq j < i$, $f(j) = g(j)$. We call such a pair (f, g) a *splitting* and the triple (i, a, b) the *index* of this splitting. We will also say that the pair (f, g) *witnesses* the index (i, a, b) in the same situation.

In subpower $\mathbf{B} \leq \mathbf{A}^n$ of an algebra with such a term operation, we define the *representation* of this subpower to be a subset $X \subseteq B$ such that for all $I \subseteq \{1, 2, \ldots, n\}$ with $|I| < k$, $proj_I(X) = proj_I(B)$ and the sets of indices witnessed in B^2 and X^2 are the same. We will say that the representation is *compact* when $|X| \leq \binom{n}{k-1}|A|^{k-1} + 2n|A|^2$.

Lemma 1. *If $X \subseteq B$ is a representation of the subpower $\mathbf{B} \leq \mathbf{A}^n$, then the subalgebra of $\mathbf{A}^n$ generated by X is $\mathbf{B}$.*

1.3 Dalmau's algorithm

The main procedure of the algorithm starts with a compact representation of A^n and imposes the constraints one by one. The main part is the procedure `Next`, which for a constraint $C_i = ((s_1, s_2, \ldots, s_{m_i}), S_i)$ and a compact representation of the subalgebra of $\mathbf{B}_{i-1} \leq \mathbf{A}^n$ produces a compact representation of the subalgebra of $\mathbf{B}_i \leq \mathbf{B}_{i-1}$ of all elements of $f \in B_{i-1}$ such that $proj_{s_1,s_2,\ldots,s_{m_i}}(f) \in S_i$.

To make the procedure `Next` work, one does a similar thing to each input constraint, by replacing S_i with $proj_{s_1}(S_i)$, then with $proj_{(s_1,s_2)}(S_i)$ and so on. This way, in each step the number of data being calculated remains small and manageable in polynomial time both in $|A|$ and n.

2 Few subpowers

2.1 A picture of some Mal'cev conditions

I don't know how to draw in LaTeX, so look at the chalkboard!

2.2 Three invariants

We introduce three spectrum-like functions for a finite algebra $\mathbf{A}$:

- $s_{\mathbf{A}}(n) = \log_2 |\mathrm{Sub}(\mathbf{A}^n)|$;
- $g_{\mathbf{A}}(n) = \max\limits_{B \in \mathrm{Sub}(\mathbf{A})} \min\limits_{\langle X \rangle = B} |X|$, the least number of elements we need to be able to generate any subalgebra of $\mathbf{A}^n$;
- $i_{\mathbf{A}}(n)$ is the maximal size of an independent subset of A^n (that is, none of its elements are in the subuniverse generated by the other elements).

Lemma 2. *The following are easy observations:*

- $g_{\mathbf{A}}(n) \leq i_{\mathbf{A}}(n) \leq s_{\mathbf{A}}(n) \leq \log_2(|A|) \cdot n g_{\mathbf{A}}(n)$.
- *If* $\mathbf{B} \in \mathcal{V}(\mathbf{A})$ *and* $|B| < \infty$*, then there exist constants* $c_i, d_i > 0$ *such that* $s_{\mathbf{B}}(n) \leq s_{\mathbf{A}}(c_1 n + d_1)$*,* $g_{\mathbf{B}}(n) \leq g_{\mathbf{A}}(c_2 n + d_2)$ *and* $i_{\mathbf{B}}(n) \leq i_{\mathbf{A}}(c_3 n + d_3)$*.*

The first item tells us that when one of the three functions is smaller than a polynomial, then all three are. In this situation, we will say that the algebra $\mathbf{A}$ has *few subpowers.*

2.3 How to prove the existence of Mal'cev terms

We saw in the tutorial by R. Willard that we need to look at the free algebra with three generators in order to prove that congruence permutability implies existence of a Mal'cev term. Instead we look at the free algebra with two free generators $\mathbf{F}_{\mathcal{V}}(x,y) =: \mathbf{F}$ in a variety $\mathcal{V}$ with permutable congruences and at its appropriate subpower. Namely, let $\mathbf{G} \leq \mathbf{F}^2$ be the subalgebra generated by the vectors $\begin{bmatrix} y \\ x \end{bmatrix}$, $\begin{bmatrix} y \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$.

We denote the projection homomorphisms in $\mathbf{F}^2$ by π_1 and π_2. Let $\eta_i := ker\pi_i \cap G^2$ be the restrictions of the kernels of these projections to G. Therefore, we have

$$\begin{bmatrix} y \\ x \end{bmatrix} \eta_1 \begin{bmatrix} y \\ y \end{bmatrix} \eta_2 \begin{bmatrix} x \\ y \end{bmatrix}.$$

Clearly, by the permutability, there must be an element $\begin{bmatrix} a \\ b \end{bmatrix} \in G$ such that

$$\begin{bmatrix} y \\ x \end{bmatrix} \eta_2 \begin{bmatrix} a \\ b \end{bmatrix} \eta_1 \begin{bmatrix} x \\ y \end{bmatrix}.$$

But, this means that $a = b = x$ and that there must be some term m such that in $\mathbf{G}$

$$m\left(\begin{bmatrix} y \\ x \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x \end{bmatrix}.$$

But this exactly means that m is a Mal'cev term, since we calculate the operations in $\mathbf{G}$ by coordinate and since $\mathbf{F}$ is a free algebra. The basic idea of this proof is most useful in many of our proofs which follow.

2.4 Cube terms and edge terms

We define a Mal'cev-style term with $2^k - 1$ variables and k equations we call the k-dimensional *cube term.* For $k = 3$ it looks like:

$$\begin{aligned} c(y,y,y,y,x,x,x) &= x \\ c(y,y,x,x,y,y,x) &= x \\ c(y,x,y,x,y,x,y) &= x. \end{aligned}$$

We also define a special kind of a cube term with all but $k+1$ many variables deleted (non-essential) which we call the k-dimensional *edge term.* For $k = 3$ we 'delete' the first, third and fifth variable and get

$$\begin{aligned} e(y,y,x,x) &= x \\ e(y,x,y,x) &= x \\ e(x,x,x,y) &= x. \end{aligned}$$

Notice that the edge term, if we would 'delete' its first variable as well would be a k-ary near-unanimity term. On the other hand, if we 'deleted' all but the first three variables it would be a Mal'cev term (with permuted variables). So, this term generalizes both near-unanimity and congruence permutability. On the other hand, the cube term implies congruence modularity (by a syntactical argument).

2.5 Few subpowers imply a cube term

Theorem 1. *Let* $\mathbf{A}$ *be a finite algebra. If for any* $c, d > 0$*,* $i_{\mathbf{A}}(cn + d) \leq n^k$ *for almost all* n*, then* $\mathbf{A}$ *has a* k*-dimensional cube term.*

Idea of the proof. By the second part of Lemma 2 we know that the assumptions hold also for the $\mathcal{V}(\mathbf{A})$-free algebra of two generators $\mathbf{F}$. Take an appropriately large n so that $i_{\mathbf{F}}(kn) < n^k$. Select a set S of n^k many $\{x, y\}$-valued tuples in $\mathbf{F}^{kn}$ in such a way that for any of them there exist k many coordinates where this is the only tuple which projects as the member of $\{x\}^k$, while all other possible projections are achieved by the other tuples. As this set can't be independent, there will be a tuple f which is in the subalgebra generated by all other tuples. Project to the 'special' k coordinates for the tuple f and by the idea for proving the existence of the Mal'cev term finish the proof.

2.6 Cube term implies edge term

Theorem 2. *Let* $\mathbf{A}$ *be a finite algebra. If* $\mathbf{A}$ *has a* k*-dimensional cube term, then* $\mathbf{A}$ *has a* k*-dimensional edge term.*

Idea of the proof. Here we use heavily the idea of section 2.3 to inductively eliminate variables one by one. For example, assume that the algebra $\mathbf{A}$ has a 5-ary term s which satisfies the equations

$$\begin{aligned} s(y,y,y,x,x) &= x \\ s(y,x,x,y,x) &= x \\ s(x,y,x,x,y) &= x. \end{aligned}$$

We desire to eliminate the second variable to obtain a 3-edge term. So assume that $\mathbf{F}(x,y)$ is the 2-generated free algebra and that $\mathbf{G} \leq \mathbf{F}^3$ is generated by vectors $\begin{bmatrix} y \\ y \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \end{bmatrix}$ and $\begin{bmatrix} x \\ x \\ y \end{bmatrix}$. If $\begin{bmatrix} x \\ x \\ x \end{bmatrix} \in G$, we are done.

So, take $\alpha = s(x,x,y,x,x)$ and

$$s\begin{pmatrix} x & x & y & x & x \\ y & x & x & y & x \\ x & y & x & x & y \end{pmatrix} = \begin{bmatrix} \alpha \\ x \\ x \end{bmatrix},$$

$$s\begin{pmatrix} x & x & y & x & x \\ y & y & y & y & y \\ x & x & x & x & x \end{pmatrix} = \begin{bmatrix} \alpha \\ y \\ x \end{bmatrix} \text{ and}$$

$$s\begin{pmatrix} x & x & y & x & x \\ y & y & y & x & x \\ x & x & x & y & y \end{pmatrix} = \begin{bmatrix} \alpha \\ x \\ y \end{bmatrix}$$

Now, just calculate

$$s\begin{pmatrix} \alpha & \alpha & \alpha & x & x \\ y & x & x & y & x \\ x & y & x & x & y \end{pmatrix} = \begin{bmatrix} x \\ x \\ x \end{bmatrix}.$$

2.7 Edge terms imply few subpowers and tractability

The proofs in this subsection are quite ingenious (*my co-authors did them*). I'll just say both are too hard for the purposes of this talk.

Lemma 3. *Let* $\mathbf{A}$ *be a finite algebra with a* k*-dimensional edge term* e*. Then* $\mathbf{A}$ *also has terms* $s(x_1, x_2, \ldots, x_k)$ *and* $p(x, y, z)$ *such that*

$$\begin{array}{rcc} p(x,y,y) & = & x \\ s(y,x,x,x,\ldots,x,x) & = & p(y,y,x) \\ s(x,y,x,x,\ldots,x,x) & = & x \\ s(x,x,y,x,\ldots,x,x) & = & x \\ & \vdots & \\ s(x,x,x,x,\ldots,x,y) & = & x. \end{array}$$

Moreover, $p(y, y, p(y, y, x)) = p(y, y, x)$.

We will call an ordered pair $(a, b) \in A^2$ such that $p(a, a, b) = b$ a *minority* pair. In other words, for the minority pair (with x evaluated as a and y as b) we get both the near-unanimity and the Mal'cev operations on this evaluation.

We define minority splitting, index, representation and so on in the same way as in Dalmau's GMM case. Then prove that

Theorem 3. *If $X \subseteq B$ is a representation of the subpower $\mathbf{B} \leq \mathbf{A}^n$, then the subalgebra of $\mathbf{A}^n$ generated by X is $\mathbf{B}$.*

Note that this means that $g_{\mathbf{A}}(n) \in O(n^{k-1})$, so $\mathbf{A}$ has few subpowers. Now we can apply exactly the same algorithm to prove that a finite idempotent algebra $\mathbf{A}$ is tractable when it has few subpowers.

THANK YOU FOR YOUR PATIENCE!