Coloured Graphs and Algebras

Andrei A. Bulatov
Simon Fraser University

What Does Identities and Types Do?

Usually we prove that if an algebra/variety does not have a term satisfying a certain identity, or if it omits certain types then

it allows some `bad' structure, and then

it is cannot be solved / cannot be defined / cannot be represented, etc.

What Do We Need to Do?

We need to be able to prove results like

If there is a term satisfying a certain identity, or algebra/variety then

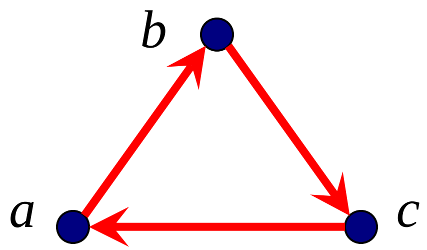
a certain algorithm works / the problem belongs to some complexity class / expressible in a certain logic, etc.

Sometimes it works:

- NU term
- 2-semilattice term
- Mal'tsev term
- GMM term
- k-cube term
- 3-Jonsson term

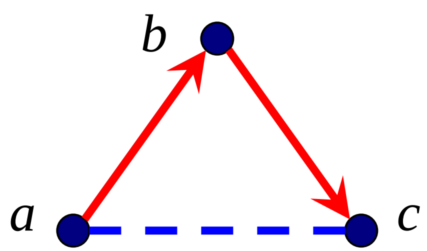
Two Algebras of Type 3

Scissors – Stone – Paper



	a	b	c
a	a	b	a
b	b	b	c
c	a	c	c

No-Name



	a	b	c
a	c	b	a
b	b	b	c
c	a	c	c

Conservative Algebras

An algebra is called conservative if every its term operation satisfies the condition

$$f(x_1,\ldots,x_n) \in \{x_1,\ldots,x_n\}$$

or, equivalently, every subset of the universe is a subalgebra

A conservative algebra does not necessarily generates a variety of conservative algebras !

See Jezek, McKenzie, Marković, Maroti for a study of varieties generated by conservative groupoids

Why Conservative Algebras

This talk: A good place to start

Most of the time:

List CSP. Given relational structures $\mathbf{A}$ and $\mathbf{B}$, and for each element $a \in \mathbf{A}$ a list $L(a)$, does there exist a homomorphism $\varphi: A \to B$ such that $\varphi(a) \in L(a)$ for all a

Graph of a Conservative Algebra (I)

Let $\mathbb{A}$ be a conservative algebra such that CSP($\mathbb{A}$) is not NP-complete

Then for any its 2-element subalgebra $\mathbb{B}$ the problem CSP($\mathbb{B}$) is not NP-complete

Therefore, for any 2-element set $\{a,b\}$ there is a term operation f such that $f_{|\{a,b\}}$ is either

a semilattice operation, or

a majority operation, or

an affine operation

Graph of a Conservative Algebra (II)

If CSP($\mathbb{A}$) is tractable, then an edge-colored graph Gr($\mathbb{A}$) can be defined on elements of $\mathbb{A}$

Theorem.

If every pair of elements is a colored edge then CSP($\mathbb{A}$) is solvable in polynomial time

Three Useful Operations

Lemma.

Let $\mathbb{A}$ be a conservative algebra such that CSP($\mathbb{A}$) is not NP-complete. Then $\mathbb{A}$ has term operations $f(x,y)$, $g(x,y,z)$, and $h(x,y,z)$ such that

- $f_{|\{a,b\}}$ is a semilattice operation on every red edge $\{a,b\}$, and a projection otherwise;
- $g_{|\{a,b\}}$ is a majority operation on every yellow edge $\{a,b\}$, a semilattice on red, and a projection on blue;
- $h_{|\{a,b\}}$ is an affine operation on every blue edge $\{a,b\}$, a semilattice on red, and a projection on yellow.

In general, f,g,h do not satisfy any useful identity

Case Study

if all edges are red then f is a 2-semilattice operation

if all edges are yellow then g is a majority operation

if all edges are blue then h is an affine (Maltsev) operation

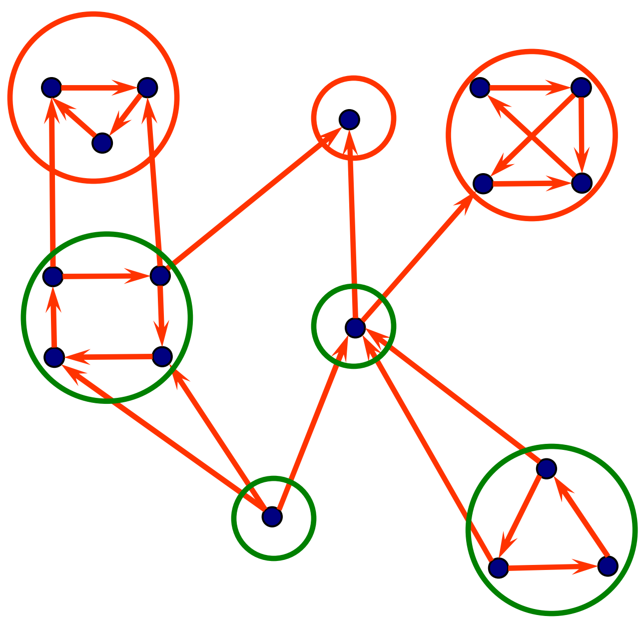
if all edges are yellow and blue

then g and h can be combined into a GMM term

Directions

If the function f is fixed then every red edge can be thought of as directed edges

Consider graph Gr'($\mathbb{A}$) obtained from Gr ($\mathbb{A}$) by removing all yellow and blue edges



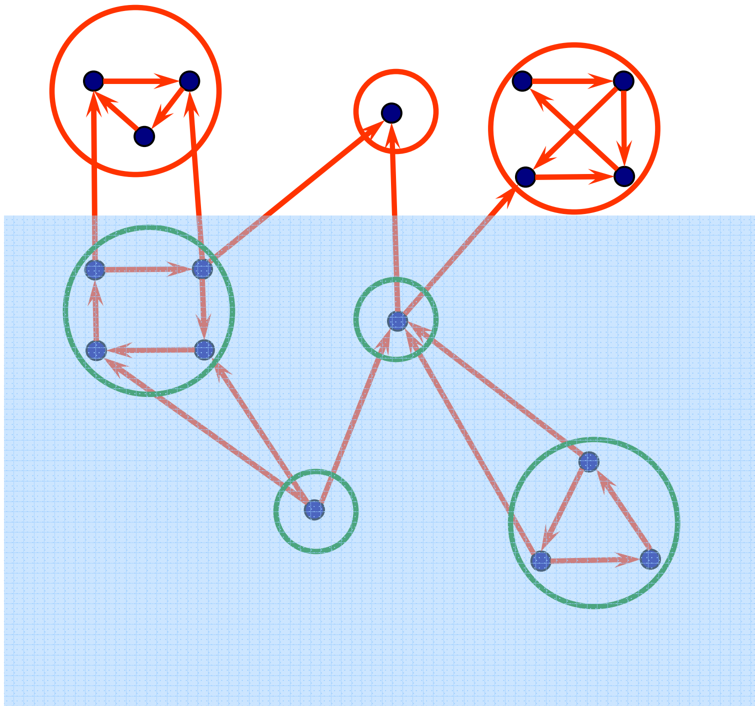
Gr'($\mathbb{A}$) is a digraph

strongly connected components

maximal strongly connected components max($\mathbb{A}$)

Algorithm (I)

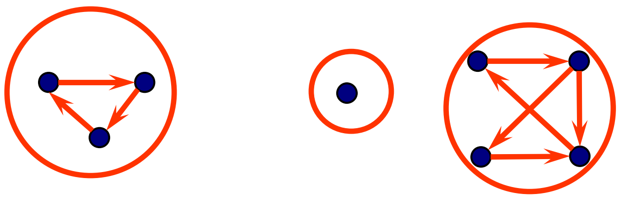
Stage 1.



If an instance of CSP($\mathbb{A}$) has a solution, it has a solution that takes only values from max($\mathbb{A}$)

Algorithm (II)

Stage 2.

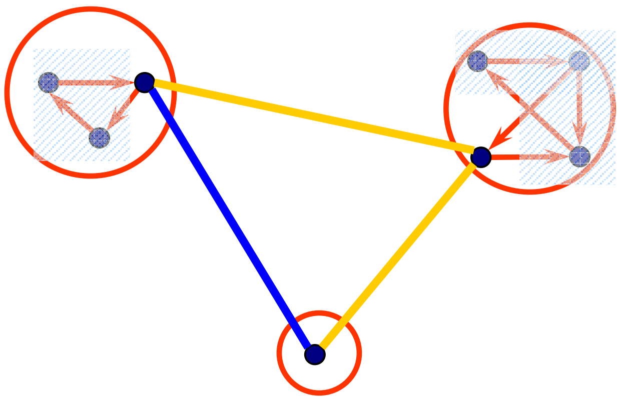


Solve the problem for each strongly connected component of max($\mathbb{A}$)

If for some of the components a solution does not exist, remove this component

Algorithm (III)

Stage 3.



Select an element from each of the remaining strongly connected components

Solve the obtained **yellow** / **blue** problem

Corollary

If Gr($\mathbb{A}$) does not have blue edges then CSP($\mathbb{A}$) has relational width 3

Generalization

Let $\mathbb{A}$ be a finite algebra satisfying the following condition

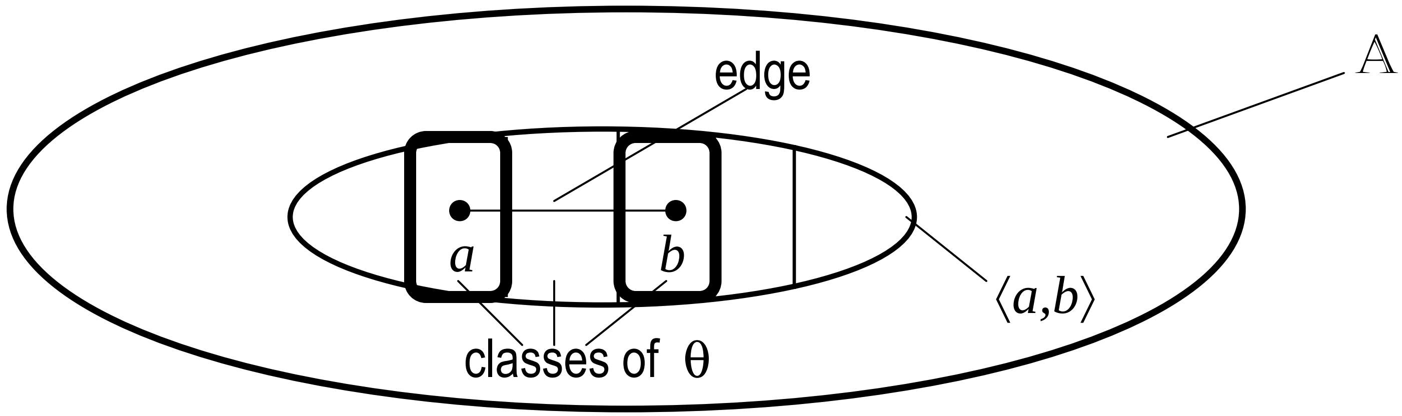
(NO-GSET) $\mathbb{A}$ is idempotent and HS($\mathbb{A}$) does not contain G-sets

---

pairs of elements $\longrightarrow$ pairs of elements modulo some congruence (not necessarily subalgebras)

local semilattice, majority, affine operations $\longrightarrow$ operations on quotient subalgebras

Generalization: the graph (I)



$Gr(\mathbb{A})$ is a graph with A as the vertex set

ab is an edge in $Gr(\mathbb{A})$ if only if there is a congruence θ of $\mathbb{B} = \langle a,b\rangle$ and a term operation f such that $f/_\theta$ is a semilattice or majority operation on $\{a/_\theta, b/_\theta\}$ or an affine operation on $\mathbb{B}/_\theta$

Generalization: the graph (II)

Theorem

For an idempotent algebra $\mathbb{A}$ the following are equivalent:

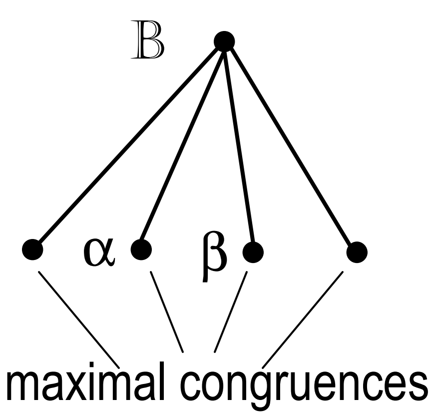
- HS($\mathbb{A}$) omits type 1
- $\mathbb{A}$ satisfies (NO-GSET)
- for every subalgebra $\mathbb{B}$ of $\mathbb{A}$, the graph Gr($\mathbb{B}$) is connected

Generalization: Colours of Edges

An edge ab is red if there are θ and f such that $f/_\theta$ is a semilattice operation on $\{a/_\theta, b/_\theta\}$

An edge ab is yellow if it is not red, and there are θ and f such that $f/_\theta$ is a majority operation on $\{a/_\theta, b/_\theta\}$

An edge ab is blue if it is not red or yellow, and there are θ and f such that $f/_\theta$ is an affine operation on $\mathbb{B}/_\theta$



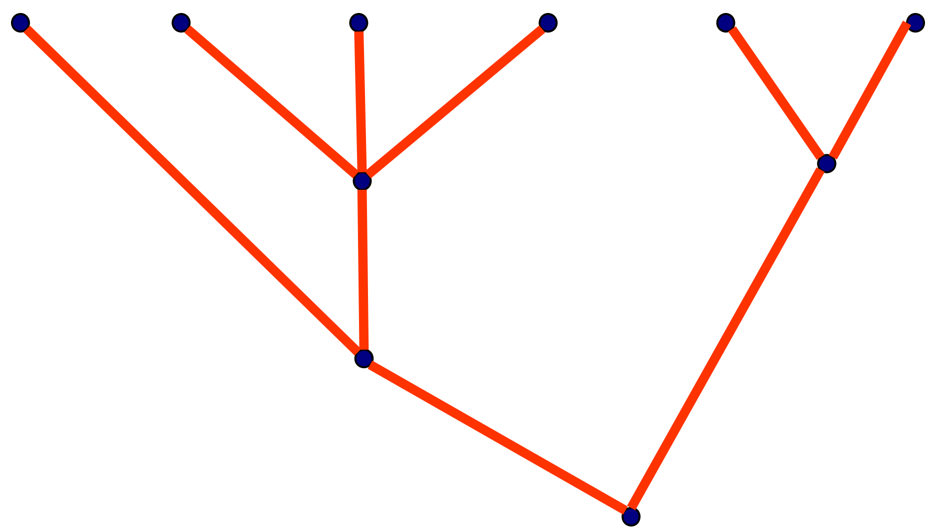
$f/_\alpha$ is semilattice on $\{a/_\alpha, b/_\alpha\}$

This prevails

there is no semilattice operation on $\{a/_\beta, b/_\beta\}$ but $g/_\beta$ is majority on this set

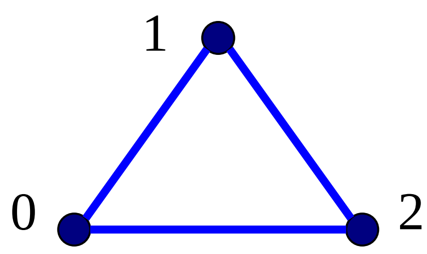
Examples (I)

Let $\mathbb{A}$ be a semilattice; then ab is an edge iff $a \cdot b \in \{a,b\}$
in this case it is a red edge



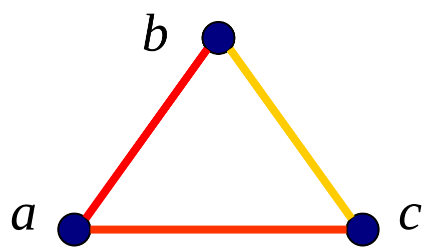
Examples (II)

Let $\mathbb{A}$ be the group $\mathbb{Z}_3$; then every pair of elements is a blue edge



Examples (III)

Let $A = \{a,b,c\}$ and $\mathbb{A} = (A;f)$ where



f	a	b	c
a	a	b	c
b	b	b	a
c	c	a	c

$g(x,y,z) = f(f(x,f(y,z)),f(f(x,y),z)$ is majority on $\{b,c\}$

Generalization: Examples (IV)

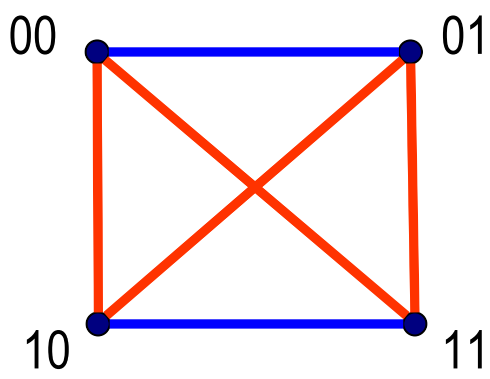
Let $\mathbb{A}$ and $\mathbb{B}$ be algebras with universe {0,1} and op-ns f, g

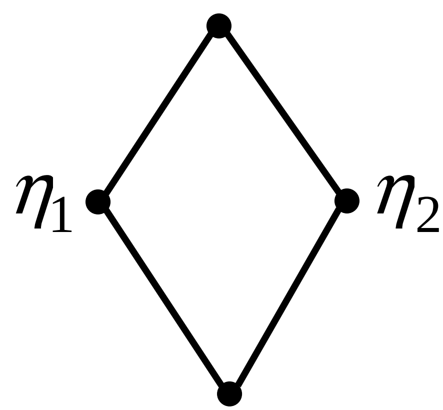
On $\mathbb{A}$ f is semilattice, g is projection

On $\mathbb{B}$ f is projection, g is affine

$\mathbb{A}\times\mathbb{B}$:

$$\langle 00,01\rangle = \{00,01\}, \langle 00,10\rangle = \{00,10\}, \langle 10,11\rangle = \{10,11\},$$
$$\langle 01,11\rangle = \{01,11\}, \langle 00,11\rangle = \langle 01,10\rangle = \mathbb{A}\times\mathbb{B}$$





Omitting Blue Edges

Theorem

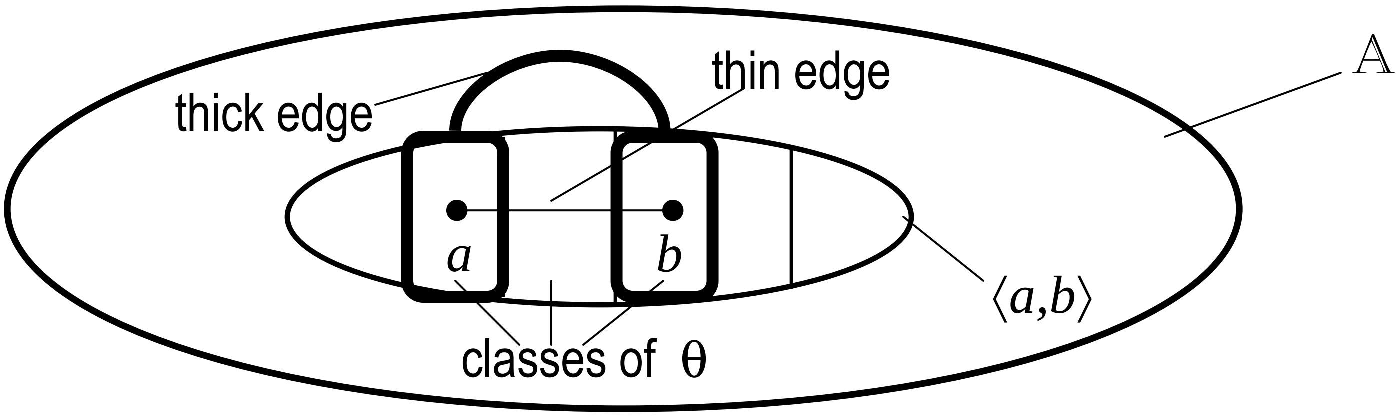
For an algebra $\mathbb{A}$ satisfying (NO-GSET) the following conditions are equivalent

- Gr($\mathbb{A}$) does not contain blue edges and Gr($\mathbb{B}$) is connected for any subalgebra $\mathbb{B}$
- HS($\mathbb{A}$) omits types 1 and 2

Corollary

If CSP($\mathbb{A}$) has bounded relational width then Gr($\mathbb{A}$) does not contain blue edges

Generalization: Thick and Thin



If ab is an edge and some congruence θ witnesses this then the set $a/_{\theta} \cup b/_{\theta}$ is called a thick edge, denoted R_{ab}

If θ can be chosen to be $_{\mathbb{B}}$ ab is called a thin edge

Adding Subalgebras

Observation

If $\mathbb{A}$' is a reduct of $\mathbb{A}$ satisfying (NO-GSET) condition then we can use $\mathbb{A}$' rather than $\mathbb{A}$

Lemma

If ab is red or yellow edge then

- $\mathbb{A}' = (A;F)$ where F = Term($\mathbb{A}$) $\cap$ Pol(R_{ab}) satisfies (NO-GSET)
- if ab is red and Gr($\mathbb{A}$) is red-connected then Gr($\mathbb{A}$') is red-connected
- if ab is red or yellow and Gr($\mathbb{A}$) is red/yellow-connected then Gr($\mathbb{A}$') is red/yellow-connected

Adding Blue Edges

Let $\mathbb{A}$ be the group $\mathbb{Z}_3$

Every pair of elements of $\mathbb{A}$ is a blue edge

Let $\mathbb{A}' = (A;F)$ where F = Term($\mathbb{A}$) $\cap$ Pol($\{0,1\}$).
It is easy to see that $\mathbb{A}'$ is a G-set

Three Useful Operations

Lemma.

Let $\mathbb{A}$ be an algebra satisfying (NO-GSET). Then $\mathbb{A}$ has term operations $f(x,y)$, $g(x,y,z)$, and $h(x,y,z)$ such that

- $f_{|\{a/_\theta, b/_\theta\}}$ is a semilattice operation on every red edge $\{a/_\theta, b/_\theta\}$, and a projection on every yellow and blue edge;
- $g_{|\{a/_\theta, b/_\theta\}}$ is a majority operation on every yellow edge $\{a/_\theta, b/_\theta\}$, a semilattice on red, and a projection on blue;
- $h_{|\{a/_\theta, b/_\theta\}}$ is an affine operation on every blue edge $\langle a,b\rangle/_\theta$ a semilattice on red, and a projection on yellow.

Red Thin Edges

Let Gr'($\mathbb{A}$) be the subgraph of Gr($\mathbb{A}$) obtained by omitting all the red edges except for thin ones

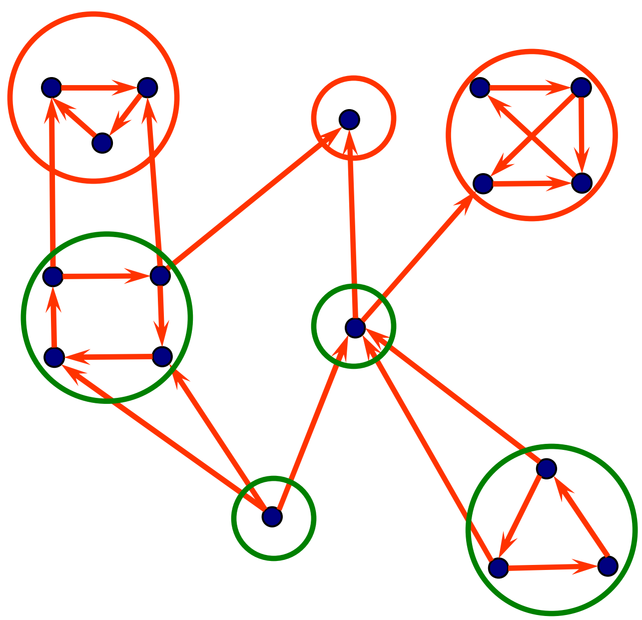
Lemma

- Gr'($\mathbb{B}$) is connected for every subalgebra $\mathbb{B}$
- If Gr($\mathbb{A}$) is red-connected then Gr'($\mathbb{A}$) is red-connected
- If Gr($\mathbb{A}$) is red/yellow-connected then Gr'($\mathbb{A}$) is red/yellow-connected

Generalization: Directions

If the function f is fixed then every red edge can be thought of as directed edges

Consider graph Gr''($\mathbb{A}$) obtained from Gr'($\mathbb{A}$) by removing all yellow and blue edges



Gr''($\mathbb{A}$) is a digraph

strongly connected components

maximal strongly connected components max($\mathbb{A}$)

Unique Red Maximal Component

Condition (maximal red component)

For every subalgebra $\mathbb{B}$ the graph Gr''($\mathbb{B}$) has a unique maximal strongly connected component

Theorem

If $\mathbb{A}$ satisfies the maximal red component condition then CSP($\mathbb{A}$) is solvable in polynomial time.

Moreover, CSP($\mathbb{A}$) has relational width 3

Applications: 2-Semilattice

Theorem

If $\mathbb{A}$ is a 2-semilattice then CSP($\mathbb{A}$) has relational width 3

$$x \cdot y = y \cdot x, \qquad x \cdot (x \cdot y) = x \cdot y$$

Every pair $a\ (a \cdot b)$ is a thin red edge

Applications: Minimal Clones (I)

C is a minimal clone if it does not have proper subclones except for the trivial one

Every minimal clone is generated by one operation

Theorem (Rosenberg)

Every minimal clone is generated by one of following operations:

- a unary operation that is either a permutation or an idempotent of the semigroup of transformation
- a binary idempotent operation that is not a projection
- a majority operation
- an affine operation
- a semiprojection

Applications: Minimal Clones (II)

Theorem

Let f be a binary operation generating a minimal clone, and $\mathbb{A} = (A; f)$. Then

(1) if Term($\mathbb{A}$) does not contain a binary commutative operation then Gr($\mathbb{A}$) is not connected

(2) if f is commutative then either $f(x,y) = \frac{1}{2}(x + y)$ or Gr''($\mathbb{A}$) satisfies the maximal red component condition

Corollary

If algebra $\mathbb{A}$ is such that Term($\mathbb{A}$) is a minimal clone generated by a binary operation. Then CSP($\mathbb{A}$) is solvable in polynomial time iff Term($\mathbb{A}$) contains a binary commutative operation; otherwise it is NP-complete