MATH 1AA3 Solutions to Written Assignment #3 April 5, 2012

1. Find the solution to the initial value problem $y^2 = 1 - y' \cos(x)$, $y(0) = 0$ as follows:

   (a) Write $y = \sum_{n=0}^{\infty} a_n x^n$ and plug into both sides of the equation. Solve for a_0, a_1, a_2, a_3.

[2] **Solution.** Since $0 = y(0) = a_0 + a_1 0 + a_2 0^2 + \cdots$, we see $\boxed{a_0 = 0}$.

$$(a_1 x + a_2 x^2 + a_3 x^3 + \cdots)^2 = 1 - (a_1 + 2a_2 x + 3a_3 x^2 + \cdots)(1 - \frac{x^2}{2} + \cdots)$$

$$0 + 0x + a_1^2 x^2 + \cdots = (1 - a_1) + (-2a_2)x + (\frac{a_1}{2} - 3a_3)x^2 + \cdots$$

Comparing coefficients:

$$0 = 1 - a_1 \quad \text{and} \quad 0 = -2a_2 \quad \text{and} \quad a_1^2 = \frac{a_1}{2} - 3a_3$$

[3] The first two equations give $\boxed{a_1 = 1}$ and $\boxed{a_2 = 0}$. The third becomes:

$$1 = \frac{1}{2} - 3a_3 \implies \boxed{a_3 = -\frac{1}{6}}.$$

   (b) Guess $y(x)$. Check that your guess is correct.

[2] **Solution.** In part (a), we found $y = x - \frac{x^3}{3!} + \cdots$. This is a familiar Taylor series, so we guess $\boxed{y = \sin(x)}$. Check:

$$(\sin(x))^2 \stackrel{?}{=} 1 - (\sin(x))' \cos(x)$$
$$\sin^2(x) \stackrel{\checkmark}{=} 1 - \cos^2(x)$$

2. (a) Using the fact that

$$\arcsin(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}},$$

find a power series expansion for $\arcsin(x)$ centred at 0. State any theorems you use.

Solution. By the Generalized Binomial Theorem,

$$(1-x^2)^{-1/2} = 1 + (-\frac{1}{2})(-x^2) + \frac{(-1/2)(-3/2)}{2!}(-x^2)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}(-x^2)^3 + \cdots$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots + \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot (2n-2)\cdot(2n)}x^{2n} + \cdots$$

[2] By Theorem 11.5.17 of Stewart (integrating power series):

$$\boxed{\arcsin(x) = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \cdots + \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot (2n-2)\cdot(2n)}\cdot\frac{x^{2n+1}}{2n+1} + \cdots}$$

(b) Write down the radius of convergence of the power series you derive in part (a). Find the interval of convergence of this power series.

[2] **Solution.** The ratio of successive terms is:

$$\frac{\cancel{1}\cdot\cancel{3}\cdot\cancel{5}\cdot\dots\cdot\cancel{(2n-1)}\cdot(2n+1)}{\cancel{2}\cdot\cancel{4}\cdot\cancel{6}\cdot\dots\cdot\cancel{(2n)}\cdot(2n+2)}\cdot\frac{x^{2n+3}}{2n+3}\div\frac{\cancel{1}\cdot\cancel{3}\cdot\cancel{5}\cdot\dots\cdot\cancel{(2n-3)}\cdot\cancel{(2n-1)}}{\cancel{2}\cdot\cancel{4}\cdot\cancel{6}\cdot\dots\cdot\cancel{(2n-2)}\cdot\cancel{(2n)}}\cdot\frac{x^{2n+1}}{2n+1}$$
$$=\frac{(2n+1)^2}{(2n+2)(2n+3)}x^2$$

which approaches x^2 as $n\to\infty$. $x^2<1 \iff -1<x<1$.

So the $\boxed{\text{radius of convergence is } 1}$.

Bonus [4] We still need to test the endpoints. At $x=1$, the series becomes:

$$1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7}+\dots+\frac{1\cdot 3\cdot 5\cdot\dots\cdot(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\dots\cdot(2n-2)(2n)}\cdot\frac{1}{2n+1}+\cdots$$

We use the limit comparison test to compare the series to $\sum\frac{1}{n^{1.5}}$:

$$\lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdot\dots\cdot(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\dots\cdot(2n-2)(2n)}\cdot\frac{1}{2n+1}\div n^{-1.5}$$
$$=\lim_{n\to\infty}\frac{n^{1.5}}{(2n+1)^{1.5}}\lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdot\dots\cdot(2n-3)\cdot(2n-1)\sqrt{2n+1}}{2\cdot 4\cdot 6\cdot\dots\cdot(2n-2)(2n)}=\frac{1}{2^{1.5}}\cdot\sqrt{\frac{2}{\pi}}$$

(The second factor comes from the square root of the reciprocal of the Wallis product: $\lim_{n\to\infty}\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\dots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}=\frac{\pi}{2}$.)

Since this limit is a positive finite number and $\sum\frac{1}{n^{1.5}}$ converges, the power series converges at $x=1$. The series for $x=-1$ is just the negation of the series for $x=1$, so converges. Therefore, the $\boxed{\text{interval of convergence is } [-1,1]}$.

(c) By choosing an appropriate value of x to plug into the power series you found in part (a), find a series that converges to $\pi/2$.

[2] **Solution.** $\pi/2=\arcsin(1)$, so :

$$\boxed{\frac{\pi}{2}=1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7}+\dots+\frac{1\cdot 3\cdot 5\cdot\dots\cdot(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\dots\cdot(2n-2)(2n)}\cdot\frac{1}{2n+1}+\cdots}$$

(d) Rederive your formula in (a) by determining the sequence $\{c_n\}$ of numbers for which

$$\sum c_n\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)^n=x$$

are equal power series.

Bonus [4] **Solution.** Writing $\arcsin(y) = c_0 + c_1 y + c_2 y^2 + c_3 y^3 + \cdots$, we have:

$$x = \arcsin(\sin(x)) = \sum c_n \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)^n$$

$$x = c_0 + c_1 x + c_2 x^2 + (c_3 - \frac{c_1}{3!})x^3 + (c_4 - 2\frac{c_2}{3!})x^4 + (c_5 - 3\frac{c_3}{3!} + \frac{c_1}{5!})x^5 + \cdots$$

Comparing the first few terms, $\boxed{0 = c_0, 1 = c_1, 0 = c_2}$. Further terms yield:

$$0 = c_3 - \frac{c_1}{3!} \implies \boxed{c_3 = \frac{1}{6}}$$

$$0 = c_4 - 2\frac{c_2}{3!} \implies \boxed{c_4 = 0}$$

$$0 = c_5 - 3\frac{c_3}{3!} + \frac{c_5}{5!} \implies \boxed{c_5 = \frac{3}{40}}$$

So $\arcsin(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots$.

3. Question 19., Problems Plus, p.783, Stewart 7th Ed. (Hint: Use the Maclaurin series for $\arctan(x)$).

[2] **Solution.** By Table 1 in Section 11.10, $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x| < 1$.
[1] In particular, for $x = \frac{1}{\sqrt{3}}$, we have

$$\frac{\pi}{6} = \arctan\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(1/\sqrt{3})^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{3}\right)^n \frac{1}{\sqrt{3}} \frac{1}{2n+1},$$

[2] so

$$\pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)3^n}\right)$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = \frac{\pi}{2\sqrt{3}} - 1}.$$

4. (a) What is the Taylor series for e^x about $x = 0$?

[1] **Solution.**

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots}$$

(b) For e^{-x}?

[1] **Solution.** Replacing x with $-x$ in (a), we get:

$$\boxed{e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots}$$

(c) The *hyperbolic sine function* is given by the power series: $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$. Using your answers to (a) and (b), write down a formula for $\sinh(x)$ that does not involve any infinite series.

[1] **Solution.** The hyperbolic sine function's power series is half the difference between the power series in (a) and (b). Therefore, $\boxed{\sinh(x) = \frac{e^x - e^{-x}}{2}}$.

(d) The *hyperbolic cosine function*, $\cosh(x)$, is the derivative of $\sinh(x)$. Write down an explicit formula for $\cosh(x)$.

[1] **Solution.**

$$\cosh(x) := (\sinh(x))' = \left(\frac{e^x - e^{-x}}{2}\right)' = \boxed{\frac{e^x + e^{-x}}{2}}$$

(e) Determine the power series expansion for $\cosh(x)$ about $x = 0$.

[1] **Solution.** Taking the derivative of the power series in (c), we get

$$\boxed{\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots}$$

(f) What is the derivative of $\cosh(x)$?

[1] **Solution.** Taking the derivative of the answer to either (d) or (e) shows $\boxed{\cosh'(x) = \sinh(x)}$.

(g) Sketch the parametric curve $x = \cosh(t), y = \sinh(t)$. Eliminate the parameter to find a Cartesian equation of the curve. Which conic (or piece of a conic) is this?

[3] **Solution.**

$$x^2 = \cosh^2(t) = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4}$$

and

$$y^2 = \sinh^2(t) = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} - 2 + e^{-2x}}{4}.$$

Subtracting the two equations yields:

$$\boxed{x^2 - y^2 = 1}.$$

This equation is a $\boxed{\text{hyperbola}}$, and the parametric curve consists of $\boxed{\text{its right half}}$.

5. Find the Taylor polynomial of smallest degree of an appropriate function about a suitable point to approximate $\sqrt{9.01}$ to within 0.00005.

[1] **Solution.** We use the Taylor series for $f(x) = \sqrt{x}$ centered at 9.

[1] The first few derivatives of $\sqrt{x}$ are $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, $f'''(x) = \frac{3}{8}x^{-5/2}$. Note: the coefficients alternate in sign.

[1] Computing $f(9) = 3$, $f'(9) = \frac{1}{6}$, $f''(9) = -\frac{1}{108}$, we find the Taylor series (at 9):

$$\sqrt{x} = 3 + \frac{1}{6}(x-9) - \frac{1}{2! \cdot 108}(x-9)^2 + \cdots \implies \sqrt{9.01} = 3 + \frac{1}{6}(0.01) - \frac{1}{216}(.01)^2 + \cdots.$$

[1] Since the series is (eventually) alternating and $\frac{1}{216}(.01)^2 < .0000005 < .00005$, we can stop before the third term.

[1] Thus the $\boxed{\text{degree one}}$ Taylor polynomial $\boxed{\sqrt{9.01} \approx 3 + \frac{1}{6}(0.01) \approx 3.00167}$ suffices.

6. (a) Using the Maclaurin series for e^x, $\sin(x)$, and $\cos(x)$, prove that $e^{ix} = \cos(x) + i\sin(x)$. Here, $i = \sqrt{-1}$ satisfies $i^2 = -1$. Read Stewart p. A63 if you get stuck.

   (b) Deduce that $e^{i\pi} = -1$.

[4] **Solution.** (a) We know that

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \frac{y^5}{5!} + \cdots$$

Setting $y = ix$ into this formula, where $x \in \mathbb{R}$ and $i^2 = -1$, yields

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \cos x + i\sin x, \end{aligned}$$

as desired.

[1] (b) Setting $x = \pi$ into the formula obtained in part (a) yields $\boxed{e^{i\pi} = \cos\pi + i\sin\pi = -1}$.

7. Which of the following are hyperbolas? For the hyperbolas, determine their foci.

   (a) $x^2 - 2x - 4y^2 = 3$

Solution.

$$\begin{aligned}(x^2-2x+1)-4y^2&=3+1\\(x-1)^2-4y^2&=4\\\frac{(x-1)^2}{2^2}-\frac{y^2}{1^2}&=1\end{aligned}$$

[3] $\boxed{\text{This is a hyperbola}}$ with foci at $(1\pm c,0)$ where $c^2=a^2+b^2=2^2+1^2=5$. Foci: $\boxed{(1\pm\sqrt{5},0)}$.

(b) $y=1/x$

Bonus [4] **Solution.** This curve is asymptotic to the coordinate axes. For it to be a hyperbola, the foci must lie on the line bisecting the angle between the asymptotes (there are two angle bisectors, but we mean the one passing through the curve). By symmetry, the foci would have to be $F_1=(f,f)$ and $F_2=(-f,-f)$. For points $P=(x,y)$ on a hyperbola, $|PF_1-PF_2|$ is constant. Looking from a point far up the vertical asymptote, the constant is seen to be the difference in vertical coordinates of the foci, or $2f$. The hyperbola would have to be:

$$\begin{aligned}\pm2f&=PF_1-PF_2\\\pm2f&=\sqrt{(x-f)^2+(y-f)^2}-\sqrt{(x+f)^2+(y+f)^2}\\\sqrt{(x-f)^2+(y-f)^2}&=\sqrt{(x+f)^2+(y+f)^2}\pm2f\\(x-f)^2+(y-f)^2&=(x+f)^2+(y+f)^2+(2f)^2\pm4f\sqrt{(x+f)^2+(y+f)^2}\\x^2-2xf+y^2-2yf+2f^2&=x^2+2xf+y^2+2yf+2f^2+4f^2\pm4f\sqrt{(x+f)^2+(y+f)^2}\\-4xf-4yf-4f^2&=\pm4f\sqrt{(x+f)^2+(y+f)^2}\\\mp(x+y+f)&=\sqrt{x^2+2xf+f^2+y^2+2yf+f^2}\\x^2+y^2+f^2+2(xy+xf+yf)&=x^2+2xf+f^2+y^2+2yf+f^2\\2xy&=f^2\\y&=\frac{f^2}{2x}\end{aligned}$$

Taking $f=\sqrt{2}$, we see that $y=\frac{1}{x}$ is indeed a $\boxed{\text{hyperbola with foci }(\sqrt{2},\sqrt{2})\text{ and }(-\sqrt{2},-\sqrt{2})}$.

(c) $y=1/x^2$

[2] **Solution.** This curve has only one axis of symmetry (the y-axis), but hyperbolas have two axes of symmetry. Therefore it is $\boxed{\text{not a hyperbola}}$.

8. Stewart Exercise 50, p 687.

[4] **Solution.** The distance from the focus (2, 1) to the directrix $x=-4$ is $2-(-4)=6$, so the distance from the focus to the vertex is $\frac{1}{2}(6)=3$ and the vertex is $(-1,\ 1)$.

Since the focus is to the right of the vertex, $p = 3$. An equation is $(y-1)^2 = 4 \cdot 3[x-(-1)]$, or $\boxed{(y-1)^2 = 12(x+1)}$.

9. Stewart Exercise 52, p 687.

[4] **Solution.** Centre is (3, 0), and $a = \frac{8}{2} = 4$, $c = 2 \Rightarrow b = \sqrt{4^2 - 2^2} = \sqrt{12} \Rightarrow$ an equation of the ellipse is $\boxed{\dfrac{(x-3)^2}{12} + \dfrac{y^2}{16} = 1}$.