MATH 1AA3 Solutions to Written Assignment #2 March 14, 2012

[2] 1. (a) To show $a_n \leq 3$ for all n, first note $a_1 \leq 3$. Now assume $a_n \leq 3$ and note $a_{n+1} = \sqrt{3a_n} \leq \sqrt{3 \cdot 3} = 3$ to conclude the inductive proof.

[2] (b) To show $a_{n+1} > a_n$ for all n, first note $a_2 = \sqrt{3} > a_1$. Now assume $a_n > a_{n-1}$ and note $a_{n+1} = \sqrt{3a_n} > \sqrt{3a_{n-1}} = a_n$ to conclude the inductive proof.

[2] (c) Since the sequence is bounded and monotonic, it converges to some value M. By the limit law for continuous functions

$$M = \lim_{n\to\infty} \sqrt{3a_{n-1}} = \sqrt{3 * \lim_{n\to\infty} a_{n-1}} = \sqrt{3M}.$$

Since $M > 1$ we have $M^2 = 3M$ or $M = 3$.

[4] 2. By definition of a sequence diverging to ∞, given any $M > 0$ there is an N such that $b_n > M$ for all $n > N$. Suppose $\{a_n\}$ is bounded below by m. Given any $M > 0$, there is an N' such that $b_n > M + m$ for all $n > N'$. For any $n > N'$, $a_n + b_n > m + (M - m) = M$, and thus $\{a_n + b_n\}$ diverges to ∞.

[6] 3. Consider the three cases $x \leq 0$, $0 < x \leq 1$, and $x > 1$. In case $x \leq 0$, the terms a_n do not converge to 0, hence the series diverges by the Test for Divergence. In both cases $0 < x \leq 1$ and $x \geq 1$ use the integral test. To check the hypotheses of the integral test, note that the function $f(z) = z^{-x}$ is continuous, positive and decreasing on $[1, \infty)$ for any $x > 0$. In case $0 < x \leq 1$, $\int_1^\infty f(z)dz$ diverges, hence the zeta series diverges. In case $x > 1$, the integral $\int_1^\infty f(z)dz$ converges, hence the zeta series converges. We conclude that the domain of the zeta function is $(1, \infty)$.

[5] 4. Note that the function $f(x) = x^{-5}$ is continuous, positive and decreasing on $[1, \infty)$ and note that $\int_n^\infty x^{-5}dx = n^{-4}/4$ for any $n > 0$. By the integral test remainder formula, $s_n + \int_{n+1}^\infty x^{-5}dx \leq s \leq s_n + \int_n^\infty x^{-5}dx$. For the width of this interval to be less than 2×10^{-3} one needs to have $(n^{-4} - (n+1)_{/}^{-4} 4 \leq 2 \times 10^{-3}$. It is sufficient to take $n = 4$ terms, whereas 3 terms is not enough. We find $s_4 + (5)^{-4}/4 = 1.0367 \leq s \leq s_4 + (4)^{-4}/4 = 1.0373$. Taking the midpoint as the estimate $\bar{s}$, we have $s \sim \bar{s} = 1.0370$ with an error less than 0.0003.

[Alternative: we also have an upper bound $s \leq s_n + \int_{n+1}^\infty x^{-5}dx + a_n$ leading to an estimate $\bar{s} = s_n + \int_{n+1}^\infty x^{-5}dx + \frac{1}{2}a_n$ with error less than $a_n/2$, which leads again to the optimal value $n = 4$.]

[3] 5. When $a_n = \frac{1}{n^{1+1/n}}$, one can compare to the divergent series with $b_n = 1/n$. We note $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n^{1/n} = e^{\lim_{n\to\infty} \ln(n)/n} = e^0 = 1$. By the limit comparison test, $\sum a_n$ diverges.

[5] 6. By the comparison test, the error $\sum_{n=11}^{\infty} a_n$ is positive and bounded above by the series $\sum_{n=11}^{\infty} \frac{1}{n^3}$, which by the integral test is bounded above by $a_{11} + \int_{11}^{\infty} \frac{dx}{x^3} = 11^{-3} + \frac{2}{11^2} = 0.0173$. Hence the error $s - s_{11}$ is positive and bounded by 0.0173. (Not asked for: On computer (MATLAB) I compute $s_{10} = 0.8325298$.)

[5] 7. We note that the function $f(x) = \sqrt{x+1} - \sqrt{x}$ is positive and decreasing on $[1, \infty)$. We also note that using l'Hospital's rule shows

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{1+1/x}-1}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{2(-1/x^2)/\sqrt{1+1/x}}{-2/x^{3/2}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}\sqrt{1+1/x}} = 0$$

By the alternating series test, one concludes that $\sum_n (-1)^n f(n)$ converges.

[5] 8. We note that the series starts with $1 - 2^{-6} + 3^{-6} - \dots$, and that $f(x) = x^{-6}$ is positive and decreasing. By the alternating series estimate, $s_{2n} \le s \le s_{2n+1}$ for any n. To make the error interval width less than 2×10^{-4} we need $a_{2n+1} < 2 \times 10^{-4}$. One finds $2n+1 = 5 > 4.135 = (2 \times 10^{-4})^{-1/6}$, so $s_4 = 0.98550 \le s \le s_5 = 0.98556$. Taking the midpoint as the estimate, we have $s \sim \bar{s} = 0.98553$ with an error less than 0.00003.

[3] 9. (a) The number of sides s_n satisfies the recurrence $s_n = 4s_{n-1}$ with $s_0 = 3$. Thus $s_n = 3 \cdot 4^n$. The lengths of the sides are $l_n = (1/3)^n$. The perimeters are $p_n = s_n * l_n = 3(4/3)^n$.

[1] (b) The sequence p_n diverges since the common ration $r = 4/3 > 1$.

(c) (Note: this problem is trickier than I thought! This is the corrected solution)
[3] Let $A_0 = \sqrt{3}/4$ be the area of the initial equilateral triangle. The area of each of the 3 smaller triangles added are $A_0 * (1/9)$. At the second step we add 4*3 triangles of area $A_0 * (1/9)$. The areas inside the snowflake after $n \ge 2$ steps are $A_n = A_0 * (1 + 3 * (1/9) + (1/3) * (4/9) + \dots + (1/3) * (4/9)^{n-1}$.This is A_0 plus a geometric series with $a = A_0/3$ and $r = 4/9$ and hence the total area in the limit is $\boxed{A = A_0[1 + 1/(3 * (1 - 4/9))] = \frac{2\sqrt{3}}{5}}$.

[4] 10. Using the hint with the two convergent geometric series $\sum b_n, \sum, c_n$ with $a = 1$ and $r = 1/2$ and $r = 1/3$ we see that the product

$$\left(1 + 2^{-1} + 2^{-2} + 2^{-3} + \dots\right)\left(1 + 3^{-1} + 3^{-2} + 3^{-3} + \dots\right)$$

generates precisely the desired series $\sum a_n$. Let the partial sums of $\sum b_n, \sum c_n$ be $\{t_n\}, \{u_n\}$ respectively. We can see that $\sum a_n$ should converge to the product $\sum b_n \cdot \sum c_n = \frac{1}{1-1/2}\frac{1}{1-1/3} = 3$.

[2] [BONUS 2 marks, because this is tricky] By some reordering of terms, we can rewrite the product series as $\sum_n d_n$ where $d_n = t_n u_n - t_{n-1}u_{n-1}$. In this case the partial sums $s_n = \sum_{i=1}^{n} d_n$ are $t_n u_n$. By the product limit law, $\lim_n t_n u_n = \lim_n t_n \cdot \lim_n u_n = 3$. Hence $\sum a_n = \sum d_n$ converges to 3.