MATH 1AA3 Solutions to Written Assignment #1 January 27, 2012

1.

$$\text{Work} = \int_R^\infty \frac{GMm}{r^2}\,dr = \lim_{t\to\infty}\int_R^t \frac{GMm}{r^2}\,dr = \lim_{t\to\infty} GMm\left[\frac{-1}{r}\right]_R^t =$$
$$= GMm \lim_{t\to\infty}\left(-\frac{1}{t}+\frac{1}{R}\right) = \frac{GMm}{R},$$

[5] where

M = mass of the earth $= 5.98\times 10^{24}$ kg, m = mass of satellite $= 10^3$ kg,
R = radius of the earth $= 6.37\times 10^6$ m, and
G = gravitational constant $= 6.67\times 10^{-11}\ \mathrm{N}\cdot\mathrm{m}^2/\mathrm{kg}$.

Therefore,

$$\text{Work} = \frac{6.67\times 10^{-11}\cdot 5.98\times 10^{24}\cdot 10^3}{6.37\times 10^6} \approx 6.26\times 10^{10}\ \mathrm{J}.$$

[5] 2. (a) $\int_{-1}^1 \frac{dx}{x}$ diverges since $\int_0^1 \frac{dx}{x}$ diverges. ($\int_0^1 \frac{dx}{x} = \lim_{t\to 0^+}[\ln(1)-\ln(t)] = \infty$.)

(b) By symmetry, $\left[\int_{-1}^{-t}\frac{dx}{x}+\int_t^1\frac{dx}{x}\right] = 0$. So $\lim_{t\to 0^+}\left[\int_{-1}^{-t}\frac{dx}{x}+\int_t^1\frac{dx}{x}\right] = \lim_{t\to 0^+} 0 = 0$.

(c)

$$\lim_{t\to 0^+}\left[\int_{-1}^{-t}\frac{dx}{x}+\int_{\alpha t}^1\frac{dx}{x}\right] = \lim_{t\to 0^+}\left[\ln|-t|-\ln|-1|+\ln|1|-\ln|\alpha t|\right]$$
$$= \lim_{t\to 0^+}\left[\ln(t)-\ln(\alpha)-\ln(t)\right] = \lim_{t\to 0^+} -\ln\alpha = -\ln\alpha.$$

(d) $\infty-\infty$ is an indeterminate form. Since parts (b) and (c) yield different numbers, we were wise to declare the integral in (a) to be divergent instead of zero, despite the rotational symmetry.

[5] 3. (a) Let $(\bar{x}_t, \bar{y}_t)$ denote the centre of mass of the lamina in question here and let A_t denote its area. Also, let $f(x) = -\ln x$. Then

$$A_t = -\int_t^1 \ln x\,dx = [x\ln x - x]_1^t = 1 - t + t\ln t,$$

and we deduce that

$$\bar{x}_t = \frac{1}{A_t}\int_t^1 x f(x)\,dx = \frac{1}{A_t}\int_1^t x\ln x\,dx = \frac{1}{A_t}\left(\left[\frac{x^2\ln x}{2}\right]_1^t - \int_1^t \frac{x}{2}\,dx\right)$$
$$= \frac{1}{A_t}\left(\frac{t^2\ln t}{2}+\left[\frac{x^2}{4}\right]_t^1\right) = \frac{1}{A_t}\left(\frac{t^2\ln t}{2}+\frac{1}{4}-\frac{t^2}{4}\right)$$
$$= \frac{2t^2\ln t + 1 - t^2}{4(1-t+t\ln t)}$$

and

$$\begin{aligned}\bar{y}_t &= \frac{1}{A_t}\int_t^1 \frac{1}{2}[f(x)]^2\,dx = \frac{1}{2A_t}\int_t^1 (\ln x)^2\,dx = \frac{1}{2A_t}[x((\ln x-2)\ln x+2)]_t^1 \\ &= \frac{1}{2A_t}\left(2-t((\ln t-2)\ln t+2)\right) = \frac{1}{2A_t}\left(2-(\ln t-2)t\ln t-2t\right) \\ &= \frac{2-(\ln t-2)t\ln t-2t}{2(1-t+t\ln t)}.\end{aligned}$$

(b) By l'Hôpital's rule, we have

$$\lim_{t\to 0^+} t^2\ln t = \lim_{t\to 0^+}\frac{\ln t}{1/t^2} = \lim_{t\to 0^+}\frac{1/t}{-2/t^3} = \lim_{t\to 0^+}\left(-\frac{t^2}{2}\right) = 0.$$

Similarly,

$$\lim_{t\to 0^+} t\ln t = 0 \quad \text{and} \quad \lim_{t\to 0^+} t(\ln t)^2 = 0.$$

Therefore, if $(\bar{x},\,\bar{y})$ is the centre of mass of R, then we have

$$\bar{x} = \lim_{t\to 0^+}\bar{x}_t = \lim_{t\to 0^+}\frac{2t^2\ln t+1-t^2}{4(1-t+t\ln t)} = \frac{1}{4}$$

and

$$\bar{y} = \lim_{t\to 0^+}\bar{y}_t = \lim_{t\to 0^+}\frac{2-(\ln t-2)t\ln t-2t}{2(1-t+t\ln t)} = 1.$$

So $(\bar{x},\,\bar{y}) = (1/4,\,1)$. Since $1 \le -\ln(1/4) = 1.386...$, the centre of mass is contained within the region R.

[10] 4. (a) For any fixed c between 0 and $s/2$, there is a square horizontal cross-section of the domical vault, two sides of which lie along $x = \pm c$ and two along $y = \pm c$. The perimeter of this square is $4 * 2c = 8c$.

To find the height of this square, use the Pythagorean theorem:

$$z = \sqrt{(s/2)^2 - c^2}.$$

The infinitesimal surface area of domical vault between the cross sections for c and $c + dc$ is:

$$\begin{aligned}\text{perimeter} \times \text{thickness} &= 8c\sqrt{1+\left(\frac{dz}{dc}\right)^2}\,dc \\ &= 8c\sqrt{1+\left(\frac{-2c}{2\sqrt{(s/2)^2-c^2}}\right)^2}\,dc \\ &= 8c\sqrt{\frac{(s/2)^2-c^2+c^2}{(s/2)^2-c^2}}\,dc \\ &= \frac{4cs}{\sqrt{(s/2)^2-c^2}}\,dc\end{aligned}$$

The total surface area is obtained by integrating, using the trig substitution $c = (s/2)\sin(\theta)$:

$$S = \int_0^{s/2} \frac{4cs}{\sqrt{(s/2)^2 - c^2}} dc$$
$$= \int_0^{\pi/2} \frac{2s^2 \sin(\theta)}{\sqrt{(s/2)^2 - (s/2)^2 \sin^2(\theta)}} ((s/2)\cos(\theta) d\theta)$$
$$= \int_0^{\pi/2} \frac{2s^2 \sin(\theta)}{(s/2)\cos(\theta)} (s/2)\cos(\theta) d\theta = 2s^2 \int_0^{\pi/2} \sin(\theta) d\theta = 2s^2.$$

(b) As in (a), the cross-section at height z is a square of sidelength $2c$ where $c = \sqrt{(s/2)^2 - z^2}$. The area of the square is

$$(2c)^2 = 4\left[(s/2)^2 - z^2\right] = s^2 - 4z^2.$$

The volume is therefore

$$V = \int_0^{s/2} (s^2 - 4z^2) dz = \left[s^2 z - \frac{4}{3} z^3\right]_{z=0}^{s/2} = \frac{s^3}{2} - \frac{4s^3}{24} = \frac{s^3}{3}.$$

The surface area S is precisely twice dV/ds. This is best understood by defining $r = s/2$. r acts like a radius of the vault in the sense that increasing its value by dr adds a vault-shaped shell around the vault, of thickness dr and surface area S. The shell has volume $dV = S * dr$. So $S = dV/dr$. By the chain rule,

$$\frac{dV}{ds} = \frac{dV}{dr}\frac{dr}{ds} = S \cdot \frac{1}{2} \Rightarrow S = 2dV/ds.$$

[7] 5. The force on a disk whose center has a depth h and whose radius is r can be computed as follows. Place the z-axis vertically, with $z = 0$ corresponding to the center of the disk. By the Pythagorean theorem, the width of the disk at z is $2\sqrt{r^2 - z^2}$ (see figure 4 of p. 554 of your textbook.) Then:

$$\text{Force} = \rho g \int_{-r}^{r} (\text{depth at } z)(\text{width at } z) dz = \rho g \int_{-r}^{r} (h - z)(2\sqrt{r^2 - z^2}) dz$$
$$= \rho g h \int_{-r}^{r} 2\sqrt{r^2 - z^2} dz - \rho g \int_{-r}^{r} z \cdot 2\sqrt{r^2 - z^2} dz$$
$$= \rho g h (\text{area of radius } r \text{ circle}) - \rho g \int_{-r}^{r} \text{odd } dz = \rho g h \pi r^2 - 0 = (\rho g) h * \pi r^2$$

(NB: Generally, the hydrostatic pressure on a vertical lamina is its area times the depth of the centroid times the weight density of the fluid, as can be seen by comparing the force and centroid integrals.)

The figure in question is obtained from removing a radius 1 disk of depth $4 = 7 - 3$ from a radius 2 disk of depth $5 = 7 - 2$. The force needed to open the tank is $9800 \cdot (5 * \pi \cdot 2^2 - 4 * \pi \cdot 1^2) = 156800\pi \approx 4.93 \times 10^5$ Newtons.

[3] 6. As x increases, the slopes decrease and all of the estimates are above the true values. Thus, all of the estimates are overestimates.

7. (a)

$$\frac{dx}{dt} = k(x-a)(x-b),\ a \neq b.$$

[10] Using partial fractions,

$$\frac{1}{(a-x)(b-x)} = \frac{1/(b-a)}{a-x} - \frac{1/(b-a)}{b-x},$$

so

$$\int \frac{dx}{(a-x)(b-x)} = \int k\,dt \Rightarrow \frac{1}{b-a}(-\ln|a-x| + \ln|b-x|) = kt + C$$
$$\Rightarrow \ln\left|\frac{b-x}{a-x}\right| = (b-a)(kt+C).$$

The concentrations $[A] = a - x$ and $[B] = b - x$ cannot be negative, so

$$\frac{b-x}{a-x} \geq 0 \quad \text{and} \quad \left|\frac{b-x}{a-x}\right| = \frac{b-x}{a-x}.$$

We now have

$$\ln\left(\frac{b-x}{a-x}\right) = (b-a)(kt+C).$$

Since $x(0) = 0$, we get

$$\ln\left(\frac{b}{a}\right) = (b-a)C.$$

Hence,

$$\ln\left(\frac{b-x}{a-x}\right) = (b-a)kt + \ln\left(\frac{b}{a}\right) \Rightarrow \frac{b-x}{a-x} = \frac{b}{a}e^{(b-a)kt}$$
$$\Rightarrow x = \frac{b[e^{(b-a)kt} - 1]}{be^{(b-a)kt}/a - 1} = \frac{ab[e^{(b-a)kt} - 1]}{be^{(b-a)kt} - a} \text{ moles}/l.$$

(b) If $b = a$, then

$$\frac{dx}{dt} = k(a-x)^2,$$

so

$$\int \frac{dx}{(x-a)^2} = \int k\,dt \quad \text{and} \quad \frac{1}{a-x} = kt + C.$$

Since $x(0) = 0$, we get

$$C = \frac{1}{a}.$$

Thus,

$$a - x = \frac{1}{kt + 1/a} \quad \text{and} \quad x = a - \frac{a}{akt+1} = \frac{a^2kt}{akt+1}\,\text{moles}\,/l.$$

Suppose $x = [C] = a/2$ when $t = 20$. Then

$$x(20) = a/2 \Rightarrow \frac{a}{2} = \frac{20a^2k}{20ak+1} \Rightarrow 40a^2k = 20a^2k + a \Rightarrow 20a^2k = a \Rightarrow k = \frac{1}{20a},$$

so

$$x = \frac{a^2t/(20a)}{1 + at/(20a)} = \frac{at/20}{1+t/20} = \frac{at}{t+20}\,\text{moles}\,/l.$$

[5] 8. $(fg)' = f'g'$, where $f(x) = e^{x^2} \Rightarrow \left(e^{x^2}g\right)' = 2xe^{x^2}g'$. Since the student's mistake did not affect the answer,

$$\left(e^{x^2}g\right)' = e^{x^2}g' + 2xe^{x^2}g = 2xe^{x^2}g'.$$

So $(2x-1)g' = 2xg$, or $\frac{g'}{g} = \frac{2x}{2x-1} = 1 + \frac{1}{2n-1} \Rightarrow \ln|g(x)| = x + \frac{1}{2}\ln(2x-1) + C \Rightarrow g(x) = Ae^x\sqrt{2x-1}$.