Math 1AA3 Test #2

Drs. Hurd, Conlon and Baker **March 6, 2012**

1. Only the Casio FX-991 calculator is permitted. Please answer all questions in the booklet provided.
2. This test has **7 questions** and **2 pages**. The total number of marks is **40**.
3. You may use the following facts without proof **(except in question 1)**:

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \\ \text{DIV} & \text{if } r \leq -1 \end{cases} \qquad \lim_{n\to\infty} \frac{1}{n^k} = \begin{cases} 0 & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \infty & \text{if } k < 0 \end{cases}$$

[2] 1. (a) Give the definition of "the sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers converges to the real number a".

[2] (b) Prove that $\lim_{n\to\infty} \frac{1}{2^n} = 0$ according to this definition.

Solution. *(a)* $\{a_n\}$ *converges to* a *if for every* $\epsilon > 0$ *there exists* N *such that whenever* $n > N$, $|a_n - a| < \epsilon$.

(b) Given $\epsilon > 0$, *take* $N > -\log_2(\epsilon)$. *Then for any* $n > N$, $|\frac{1}{2^n} - 0| = 2^{-n} < 2^{-N} < 2^{\log_2 \epsilon} = \epsilon$. *So* $\frac{1}{2^n} \to 0$.

[4] 2. In each of the following examples, show whether or not the sequence converges to a real number, and find the limit when the sequence converges. Name any test or rule that you use.

(a)

$$\left\{ \frac{(-1)^n + 1}{n} \right\},$$

(b)

$$\left\{ \frac{\sqrt{n} + 7}{\sin(n) + 3\sqrt{n}} \right\}.$$

Solution. *(a)*

$$0 \leq \frac{(-1)^n + 1}{n} \leq \frac{2}{n}$$

Since $\lim_{n\to\infty} 0 = 0$ *and* $\lim_{n\to\infty} \frac{2}{n} = 2\lim_{n\to\infty} \frac{1}{n} = 2 \cdot 0 = 0$, *the Sandwich Theorem tells us*

$$\lim_{n\to\infty} \frac{(-1)^n + 1}{n} = \boxed{0}$$

(b) Since $-1 \leq \sin(n) \leq 1$*:*

$$\frac{\sqrt{n}+7}{1+3\sqrt{n}} \leq \frac{\sqrt{n}+7}{\sin(n)+3\sqrt{n}} \leq \frac{\sqrt{n}+7}{-1+3\sqrt{n}}$$

As:

$$\lim_{n\to\infty} \frac{\sqrt{n}+7}{1+3\sqrt{n}} = \lim_{n\to\infty} \frac{1+\frac{7}{\sqrt{n}}}{\frac{1}{\sqrt{n}}+3} = \frac{1+7\cdot 0}{1\cdot 0+3} = \frac{1}{3}$$

and:

$$\lim_{n\to\infty} \frac{\sqrt{n}+7}{-1+3\sqrt{n}} = \lim_{n\to\infty} \frac{1+\frac{7}{\sqrt{n}}}{-\frac{1}{\sqrt{n}}+3} = \frac{1+7\cdot 0}{-1\cdot 0+3} = \frac{1}{3}$$

the sandwich theorem tells us

$$\lim_{n\to\infty} \frac{\sqrt{n}+7}{\sin(n)+3\sqrt{n}} = \boxed{\frac{1}{3}}$$

[Alternative Solution: Using Limit Laws,

$$\frac{\sqrt{n}+7}{\sin(n)+3\sqrt{n}} = \frac{1+7/\sqrt{n}}{\sin(n)/\sqrt{n}+3} \to \frac{1+0}{0+3}$$

since $\lim_{n\to\infty} \sin(n)/\sqrt{n} = 0$ *by the Squeeze Theorem and* $\lim_{n\to\infty} 1/\sqrt{n} = 0$*.]*

[6] 3. For each of the following statements, say whether it is true or false. For those statements that are false, give a reason or a counterexample.

Solution. *(a) If* $a_n > 0$ *for all* $n \in \mathbb{N}$, *and* $a_n \longrightarrow a$, *then* $a > 0$.

(b) The series $\sum_{n=1}^{\infty} a_n$ *converges if its sequence of partial sums* (s_n) *is bounded.*

(c) $\sum_{n=p}^{\infty} ar^n = \frac{ar^p}{1-r}$ *provided that* $|r| < 1$.

(a) $\boxed{FALSE}$. *Counterexample:* $1/n > 0$ *for all* n *but* $1/n \longrightarrow 0$.

(b) $\boxed{FALSE}$. *Counterexample:* $\sum (-1)^n$ *diverges.*

(c) $\boxed{TRUE}$.

[3] 4. (a) Determine whether $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ converges or diverges, by use of an appropriate test. In the case the series converges, find its sum.

[3] (b) Use the Integral test to determine the values of p for which the following series is convergent.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

Solution. *(a) This is a telescoping series.*

$$\begin{aligned} s_n &= \sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \ln(n) - \ln(n+1) \\ &= \ln(1) - \ln(2) + \ln(2) - \ln(3) + \ln(3) - \ln(4) + \cdots + \ln(n) - \ln(n+1) \\ &= \ln(1) - \ln(n+1) = -\ln(n+1) \end{aligned}$$

As $n \to \infty$, $\ln(n+1) \to \infty$, so the series $\boxed{\textit{diverges}}$.

(b)

$$\int_e^{\infty} \frac{dx}{x(\ln(x))^p} = \int_1^{\infty} \frac{du}{u^p} = \frac{u^{1-p}}{1-p}\bigg|_1^{\infty}$$

converges if and only if $p > 1$.
By the Integral Test, the series converges if and only if $\boxed{p > 1}$.

[6] 5. Solve the following initial value problems on the indicated intervals.

(a) $x\frac{dy}{dx} = y$, $y(1) = 2$. Give an answer valid for all x.

(b) $x\frac{dy}{dx} = x + y$, $y(1) = 2$. Give an answer valid for $x > 0$.

Solution. *(a)*

$$\begin{aligned} \frac{dy}{y} &= \frac{dx}{x} \\ \int \frac{dy}{y} &= \int \frac{dx}{x} \\ \ln(y) &= \ln(x) + C \\ y &= e^{\ln(x)+C} = Ax \quad \text{where } A = e^C \end{aligned}$$

Using $y(1) = 2$, we see $2 = A(1)$, so $A = 2$. The solution is $\boxed{y{=}2x}$.

(b)

$$x\frac{dy}{dx} = x + y \implies \frac{dy}{dx} - \frac{1}{x}y = 1.$$

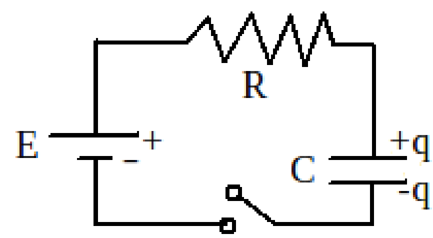
The integrating factor is $e^{\int -dx/x} = e^{-\ln(x)} = 1/x$.

$$\begin{aligned} \frac{1}{x}y' - \frac{1}{x^2} &= \frac{1}{x} \\ (y/x)' &= \frac{1}{x} \\ y/x &= \int \frac{1}{x} = \ln(x) + C \\ y &= x\ln(x) + Cx \end{aligned}$$

Since $y(1) = 2$, $2 = 1\ln(1) + C \cdot 1 = C$, *so the solution is* $\boxed{y = x\ln(x) + 2x}$.

6. An RC-circuit (resistor-capacitor circuit) consists of a power source of voltage E, a capacitor of (constant) capacitance C, and a resistor of (constant) resistance R configured as in the diagram. The capacitor starts out with charge $q = 0$ at time $t = 0$, when the switch is closed. The circuit then obeys the differential equation:

$$E = \frac{q}{C} + R\frac{dq}{dt}$$



[5] (a) Determine the charge q on the capacitor as a function of time if E is a positive constant. (Your answer should give q in terms of E, R, C, t.)

[2] (b) What happens to the charge as time approaches ∞? (Again, assume E is positive constant.)

Solution. *(a)* Solution 1: As Separable Equation

$$\begin{aligned} E &= \frac{q}{C} + R\frac{dq}{dt} \\ EC - q &= RC\frac{dq}{dt} \\ \int \frac{RC\,dq}{q - EC} &= \int -dt \\ RC\ln(q - EC) &= -t + k \quad k\text{=const of integration} \\ \ln(q - EC) &= -t/(RC) + k_2 \quad (\text{where } k_2 = k/(RC)) \\ q - EC &= e^{-t/(RC)+k_2} = Ae^{-t/(RC)} \quad \text{where } A = e^{k_2} \\ q &= EC - Ae^{-t/(RC)} \end{aligned}$$

At $t=0$, $0=q=EC-Ae^0=EC-A$, *so* $A=EC$. *Thus* $\boxed{q=EC(1-e^{-t/(RC)})}$.

Solution 2: As Linear Equation

$$
\begin{aligned}
E &= \frac{q}{C}+R\frac{dq}{dt}\\
\frac{dq}{dt}+\frac{q}{RC} &= \frac{E}{R} \quad \textit{integrating factor} = e^{\int \frac{dt}{RC}} = e^{t/(RC)}\\
e^{t/(RC)}\frac{dq}{dt}+e^{t/(RC)}\frac{q}{RC} &= \frac{E}{R}e^{t/(RC)}\\
(e^{t/(RC)}q)' &= \frac{E}{R}e^{t/(RC)}\\
e^{t/(RC)}q &= \int Ee^{t/(RC)}dt/R = ECe^{t/(RC)}+k \quad k{=}\textit{const of integration}\\
q &= EC+ke^{-t/(RC)}
\end{aligned}
$$

When $t=0$, $0=q=EC+ke^0=EC+k$, *so* $k=-EC$. *Thus* $\boxed{q=EC(1-e^{-t/(RC)})}$.

(b) As $t\to\infty$, $e^{-t/(RC)}\to e^{-\infty}=0$. *So* $\boxed{q\to EC}$.

7. Consider the sequence $\{s_n\}_{n=1}^{\infty}$ where $s_n=1-\frac{1}{(n+1)^2}, n=1,2\ldots$.

[2] (a) Find the series $\sum_{n=1}^{\infty}a_n$ that has $\{s_n\}$ as its sequence of partial sums.

[2] (b) Determine if the series $\sum_n a_n$ converges, and if it does, compute its sum.

[3] (c) What is the minimum number n of terms needed so that the remainder $R_n=s-s_n$ is less than 10^{-4}?

Solution. *(a)* $a_n=s_n-s_{n-1}=(1-\frac{1}{(n+1)^2})-(1-\frac{1}{n^2})=\frac{1}{n^2}-\frac{1}{(n+1)^2}=\frac{2n+1}{(n(n+1))^2}$, *so the series whose partial sum is* s_n *is* $\sum_{n=1}^{\infty}\frac{2n+1}{(n(n+1))^2}$.

(b) $s_n\longrightarrow 1$ *as* $n\longrightarrow\infty$, *therefore the series* $\boxed{\textit{converges}}$ *and its sum is* $\boxed{1}$.

(c) $R_n=s-s_n=\frac{1}{(n+1)^2}<10^{-4}$ *if* $(n+1)^2>10^4$, *i.e., if* $n>99$, *so the minimum number of terms is* $\boxed{n=100}$.

END OF TEST