Math 1AA3 Test #1 Solutions

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1. True/False questions worth 2 marks each. No part marks are awarded.

[2] (a) The function $x(t) = 2\sin 2t$ is a solution of the differential equation $\frac{d^2x}{dt^2} = -4x$.

Solution. $\boxed{\textbf{\textit{True}}}$. *Both sides are equal when* $x(t) = 2\sin(2t)$ *is substituted.* $x'(t) = 4\cos(2t)$ *so the left-hand side of the differential equation is* $x''(t) = -8\sin(2t)$. *The right-hand side is also* $-4 \cdot 2\sin(2t) = -8\sin(2t)$. □

[2] (b) The improper integral $\int_1^\infty \frac{dx}{\sqrt{x^3+x^{3/2}}}$ is convergent.

Solution. $\boxed{\textbf{\textit{True}}}$. *The integrand is:*

$$\frac{1}{\sqrt{x^3+x^{3/2}}} \le \frac{1}{\sqrt{x^3}} = \frac{1}{x^{1.5}}$$

By the p-test, $\int_1^\infty dx/x^{1.5}$ *converges (since* $1.5 > 1$*). The Comparison Theorem then implies the integral in question converges.* □

[6] 2. McMaster's new beanie cap is designed to be the surface obtained by taking the circular arc $y = \sqrt{1-x^2}$ lying over the x-interval $[-1/2, 1/2]$ and revolving it around the y-axis. Compute the surface area of the beanie.

Solution. *The derivative is* $y' = -2x \cdot \frac{1}{2}(1-x^2)^{-1/2} = -x/\sqrt{1-x^2}$, *so* $(y')^2 = x^2/(1-x^2)$.

Since the distance from (x, y) *to the* y*-axis is* x *(not* $f(x)$*) we must compute the surface area* S *by:*

$$S = \int_0^{1/2} 2\pi x\sqrt{1+(y')^2}dx = 2\pi\int_0^{1/2} x\sqrt{1+\frac{x^2}{1-x^2}}dx = 2\pi\int_0^{1/2} x\sqrt{\frac{1-x^2+x^2}{1-x^2}}dx$$

$$= \pi\int_0^{1/2}\frac{2xdx}{\sqrt{1-x^2}} = \pi\int_1^{3/4}\frac{-du}{\sqrt{u}} = \pi\int_{3/4}^1 u^{-1/2}du = 2\pi\sqrt{u}\Big|_{3/4}^1 = 2\pi(1-\frac{\sqrt{3}}{2}) = \boxed{\pi(2-\sqrt{3})}$$

where the second line uses the substitution $u = 1-x^2, du = -2xdx$.

Note: the limits of integration are $[0, 1/2]$, *not* $[-1/2, 1/2]$, *since the* $[0, 1/2]$ *portion sweeps out the entire beanie area.*

Alternative solution: *Write* $x = g(y) = \sqrt{1-y^2}$, $\sqrt{1+(g'(y))^2} = \sqrt{y^2/(1-y^2)}$ *and find*

$$S = \int_{\sqrt{3}/2}^{1} 2\pi\sqrt{1-y^2}\sqrt{y^2/(1-y^2)}dy = 2\pi\int_{\sqrt{3}/2}^{1} dy = 2\pi(1-\frac{\sqrt{3}}{2}) = \boxed{\pi(2-\sqrt{3})}$$

□

[5] 3. A cylindrical tank has radius 1 m and length 7 m and is laid horizontally on its long side. It is filled completely with milk with density $1000kg/m^3$ and the acceleration of gravity is $g = 9.8m/s^2$. Compute the total work needed to empty half the tank, if pumped from the top of the tank.

Solution. *Consider the horizontal cross-section a distance* y *above the horizontal diagonal. By the Pythagorean Theorem, it is a rectangle of dimensions* $2\sqrt{1-y^2}$ *by* 7. *It needs to be raised a distance* $1-y$. *The work done on a thin rectangular slab is thus*

$$1000\cdot 9.8\cdot(2\sqrt{1-y^2})\cdot 7\cdot(1-y)dy = 137200(1-y)\sqrt{1-y^2}dy.$$

The total work in emptying the top half is:

$$W = 137200\int_0^1 (1-y)\sqrt{1-y^2}dy = 137200\left(\int_0^1 \sqrt{1-y^2}dy - \int_0^1 y\sqrt{1-y^2}dy\right)$$

The first integral is 1/4 the area of the unit circle, or $\pi/4$.

The second integral is computed by the substitution $u = 1-y^2, du = -2ydy$:

$$\int_0^1 y\sqrt{1-y^2}dy = -\frac{1}{2}\int_1^0 \sqrt{u}du = \frac{1}{2}\int_0^1 \sqrt{u}du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_0^1 = \frac{1}{3}$$

Thus $W = \boxed{137200(\frac{\pi}{4}-\frac{1}{3}) \approx \textit{sixty thousand Joules}}$. □

[6] 4. Express each of the following improper integrals in terms of limits of proper integrals. Determine in each case whether the integral is divergent or convergent, justifying your answer. When the integral is convergent compute its value.

(a) $\int_0^\infty \frac{dx}{1+x^2}$;

Solution.

$$\int_0^\infty \frac{dx}{1+x^2} = \boxed{\lim_{t\to\infty}\int_0^t \frac{dx}{1+x^2}} = \lim_{t\to\infty}(\arctan(t) - \arctan(0)) = \pi/2 - 0 = \boxed{\frac{\pi}{2}}$$

The integral $\boxed{\textbf{\textit{converges}}}$. □

(b) $\int_0^4 \frac{dx}{1-x^2}$.

Solution. *The denominator is zero when $x = 1$. So:*

$$\int_0^4 \frac{dx}{1-x^2} = \boxed{\lim_{t\to 1^-} \int_0^t \frac{dx}{1-x^2} + \lim_{t\to 1^+} \int_t^4 \frac{dx}{1-x^2}}$$

Using partial fractions, the first summand is

$$\lim_{t\to 1^-} \int_0^t \frac{dx}{1-x^2} = \lim_{t\to 1^-} \frac{1}{2} \int_0^t \frac{1}{1+x} + \frac{1}{1-x} dx = \lim_{t\to 1^-} \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)\Big|_0^t = \infty.$$

So $\int_0^1 \frac{dx}{1-x^2}$ diverges. Therefore $\int_0^4 \frac{dx}{1-x^2}$ $\boxed{\textbf{diverges}}$.
Alternative Solution: Use the comparison test: $\int_0^1 \frac{dx}{2(1-x)}$ diverges and $0 \le \frac{1}{2(1-x)} \le \frac{1}{(1-x)(1+x)} = \frac{1}{1-x^2}$, hence $\int_0^1 \frac{dx}{1-x^2}$ $\boxed{\textbf{diverges}}$. □

[5] 5. Find the centre of mass of the lamina of uniform density that lies between the lines $x = 0$, $x = 1$, above the x-axis and below the graph of the function $y = f(x) = \sqrt{1-x^2}$.

Solution. *The region in question is the portion of the unit circle in the first quadrant. The y-coordinate of the centroid (center of mass) is:*

$$\overline{y} = \frac{\int_0^1 \frac{1}{2} f(x)^2 dx}{\text{Area of Quarter Circle}} = \frac{\int_0^1 \frac{1}{2}(1-x^2)dx}{\pi/4} = \frac{2}{\pi}\left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{2}{\pi}\cdot(1-1/3) = \boxed{\frac{4}{3\pi}}$$

By symmetry about the line $y = x$, the x-coordinate of the centroid (geometric center) must be $\boxed{\overline{x} = \overline{y} = 4/(3\pi)}$.

□

6. Solve the following differential equations for the function $y(t)$, with the given initial conditions. In both cases, make sure you determine the interval on which the solution is continuous.

[4] (a) $y' = ye^t$, $y(0) = e$;

Solution. *Separable DE:*

$$\begin{aligned} \int \frac{dy}{y} &= \int e^t dt \\ \ln|y| &= e^t + C \\ |y| &= e^{Ae^t}, A = e^C \end{aligned}$$

Sub in the values $y = e, t = 0$ to find $A = 1$. Hence the solution is $\boxed{y(t) = e^{e^t}}$, valid for all t. □

[4] (b) $t^2y' + \frac{ty}{\ln t} - 2t = 0,\ y(2) = 0$

Solution. *The linear DE in standard form is* $y' + \frac{1}{t\ln t}y = 2/t$. *First find the integrating factor*

$$I(t) = e^{\int dt/(t\ln t)} = e^{\ln(\ln t)} = \ln t$$

where the integral is done using the substitution $u = \ln t, du = dt/t$. *Multiplication of the DE by* $I(t) = \ln t$ *leads to the DE* $[(\ln t)y]' = 2\ln t/t$ *which integrates to:*

$$(\ln t)y = 2\int \ln t dt/t = (\ln t)^2 + C,$$

again using the substitution $u = \ln t, du = dt/t$. *Sub in the values* $y = 0, t = 2$ *to find* $C = -(\ln 2)^2$. *The solution is* $\boxed{y(t) = \ln t - (\ln 2)^2/\ln t}$ *and is valid on the interval* $(1, \infty)$. □

7. Suppose the world population $P(t)$ follows the differential equation

$$\frac{dP}{dt} = k\ln(M/P)P$$

with today's population $P(0) = 7 * 10^9$ and parameter values $k = 0.029/year, M = 10^{10}$.

[4] (a) Solve for the world population P as a function of time t.

Solution. *Separable DE leads to:*

$$\begin{aligned}
\int \frac{dP}{P\ln(M/P)} &= k\int dt \\
\int \frac{du}{-u} &= kt + C, \text{ where } u = \ln(M/P), du = -dP/P \\
-\ln(|\ln(M/P)|) &= kt + C \\
|\ln(M/P)| &= Ae^{-kt}, \quad A = e^{-C} \\
P(t) &= Me^{-Ae^{-kt}},
\end{aligned}$$

Sub in the values $P = P(0), t = 0$ *to find* $e^{-A} = P(0)/M = 0.7$. *Conclude* $\boxed{P(t) = M(0.7)^{e^{-kt}}}$.

□

[2] (b) Use this formula to compute the projected world population on February 7, 2112.

Solution. $\boxed{P(1) = 10^{10} * (.7)^{e^{-.029}} = 7.07173 * 10^9}$ □

END OF TEST