Math 1AA3 Practice Test #2

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1. Only the Casio FX-991 calculator is permitted. Please answer all questions in the booklet provided. Please write in pen, not pencil.

2. You may use the following facts without proof:

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \\ \text{DIV} & \text{if } r \leq -1 \end{cases} \qquad \lim_{n\to\infty} \frac{1}{n^k} = \begin{cases} 0 & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \infty & \text{if } k < 0 \end{cases}$$

1. For each of the following statements, say whether it is true or false. For those that are false, give a reason or a counterexample.
   (a) The sequence $(a_n) \longrightarrow 0$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges.
   (b) If $\{a_n\}$ is divergent, then $\{|a_n|\}$ is divergent.
   (c) If $\lim_{n\longrightarrow\infty} |a_n|$ exists, then $\lim_{n\longrightarrow\infty} a_n$ exists.
   (d) If $\lim_{n\longrightarrow\infty} a_n = L$, then the telescoping series $\sum_{n=1}^{\infty}(a_{n+1} - a_n)$ converges and has sum $L - a_1$.
   (e) If $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ must also converge.

2. (a) Give the definition of "the sequence $\{a_n\}$ of real numbers tends to infinity".
   (b) Prove that $\lim_{n\to\infty} \ln(\sqrt{n}) = \infty$ according to this definition.

3. Let p be any positive integer. Consider the sequence $\{b_n\}_{n=1}^{\infty}$ with $b_n = n^p e^{-n}$.
   (a) Show that $\lim\limits_{n\to\infty} b_n = 0$.
   (b) For $p = 2$, use the integral test to show that $\sum b_n$ converges.
   (c) For $p = 2$, use the limit comparison test to show that $\sum b_n$ converges.
   (d) How many terms would be needed to estimate $\sum n^{10} e^{-n}$ to within a tolerance of ± 1.

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{\ln n}$ converges or diverges. In the case that the series converges, find its sum.

5. (a) Using the Integral test, determine the values of p for which the following series is convergent.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$$

   (b) Consider the series with $p = 4$. How many terms of this series must we add to approximate the sum with error less than 0.001? You may assume that $\ln x \leq x - 1$ for $x > 0$.

6. Consider the series $\sum_{n=1} \frac{(-1)^n}{\ln(n+1)}$.

   (a) Show that this series converges.

   (b) If the sum of the series is denoted s, find both an upper and lower bound on the remainder $R_{99} = s - s_{99}$. Is the sign of R_{99} positive or negative?

   (c) What is the smallest number N such that $0 \leq R_N \leq 10^{-1}$?

7. Consider the family of curves $\{y^2 = x + C\}$ (a different curve for each value of C).

   (a) Find a single differential equation to which all the above curves are solutions.

   (b) Find a differential equation satisfied by the orthogonal trajectories. (Hint: perpendicular lines have negative reciprocal slopes.)

   (c) Solve this differential equation to determine all orthogonal trajectories to $\{y^2 = x + C\}$.

8. The hormone concentration in the bloodstream $h(t)$ has been modelled by a differential equation

$$\frac{dh}{dt} = -\frac{R}{V}\left(\frac{h}{k+h}\right)$$

   where R, V, k are parameters.

   (a) Solve this differential equation to find the relationship between h and t.

   (b) Solve the initial value problem with $h(0) = h_0$.

   (c) What is the long time behaviour of $h(t)$?

END OF TEST