Mathematics 1AA3 Practice Exam (subject to modification)

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Approximate Duration of Practice Exam: 3 hours

Name ______________________________

Student I.D. ______________________________

THIS PRACTICE EXAMINATION HAS 10 QUESTIONS AND 3 PAGES FOR YOUR ENTERTAINMENT.

ENJOY!

- The university approved Casio FX-991 calculator is allowed.
- Total marks: [**XX**]. Please answer all questions on the examination paper.
- The following formulas may prove to be useful:
  - Area of surface of revolution (revolving the curve $y = f(x), a \leq x \leq b$ about x-axis):
  $$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}dx \ .$$
  - Work in moving with a force $F(x)$ from $x = a$ to $x = b$:
  $$W = \int_a^b F(x)dx \ .$$
  - Indefinite integral (for $a \neq 0$):
  $$\int \sqrt{a^2 + x^2}\ dx = \frac{x}{a}\sqrt{a^2 + x^2} + \frac{a^2}{2}\ln(x + \sqrt{a^2 + x^2}) + C$$
  - **Taylor's Inequality:** If $|f^{n+1}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality
  $$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1}, \quad \text{for } |x - a| \leq d \ .$$

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1. True/False questions worth 2 marks each. Circle the correct response. No part marks are awarded.

   (a) If the parametric curve $x = f(t), y = g(t)$ has $g'(1) = 0$ then it has a horizontal tangent at $t = 1$.
   T / F

   (b) The differential equation $y' = 5 + 3xy + 3y + 5x$ is separable.
   T / F

   (c) If a series converges, then the sequence of its partial sums also converges.
   T / F

   (d) The improper integral $\int_{-1}^{1} \frac{dx}{|x|}$ converges.
   T / F

2. A spherical tank of diameter $6m$, filled with water, is buried below the surface with its centre at a depth of $7m$. Compute the work needed to pump the entire volume of water to the surface. You should take the density of water to be 1000 $kg\ m^{-3}$ and the acceleration of gravity to be $g = 9.8\ m\ s^{-2}$.

3. In each of the following examples, show whether or not the sequence converges to a real number, and find the limit when the sequence converges. Name any test or rule that you use. You can assume that $\{r^n\} \longrightarrow 0$ if $|r| < 1$, $(r^n) \longrightarrow \infty$ if $r > 1$ and $\left(\frac{1}{n^k}\right) \longrightarrow 0$ when $k > 0$.

   (a)
   $$\left\{\frac{3^n + 5^n}{4^n}\right\}$$

   (b)
   $$\left\{\frac{n!}{4n^n}\right\}$$

4. The population of bacteria in a culture grows at a rate that is proportional to the number present. After 2 hr there are 800 bacteria present. After 4 hr there are 3200 bacteria present. How many bacteria were there initially?

5. The curve $y = e^{-x/2}, 0 \leq x \leq 5$ is rotated about the x-axis, to give the shape of a long trumpet. Compute the area of the resulting surface. (Needs the integral $\int \sqrt{1 + u^2} du$).

6. (a) Give the precise (M, N) definition of the statement "the sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers diverges to infinity".

   (b) Prove that $\{n^2\} \longrightarrow \infty$ according to this definition.

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7. Solve the differential equations:

   (a) $xy' + (2x+1)y = xe^{2x}$

   (b) $xy' = y^2 \ln x$

8. (a) Using the Integral test, determine the values of p for which the following series is convergent.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

   (b) Determine the convergence of the following series by using an appropriate test: $\sum_{n=1}^{\infty} \frac{n^3}{n!}$

   (c) Determine the convergence of the following series by using an appropriate test: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n^2 - e^{-n}}}{\sqrt{n^2 + e^{-n}}}$

9. Consider the parametric curve $x = t^2 - t, y = \sqrt{t}$.

   (a) Sketch the curve for the parameter values $0 \le t \le 4$, showing the location of all intercepts, vertical and horizontal tangents.

   (b) Compute the area of the closed shape between the curve and the y-axis.

10. Consider the series $\sum_{n=1} a_n$ with $a_n = \frac{(-1)^n}{\ln(n+1)}$.

   (a) Show that this series converges.

   (b) If the sum of the series is denoted s, find a bound on the remainder $R_{99} = s - s_{99}$. Is the sign of R_{99} positive or negative?

   (c) What is the smallest number N such that $0 \le R_N \le 10^{-2}$?

END OF TEST