Credit risk modelling using time-changed Brownian motion

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Abstract

Motivated by the interplay between structural and reduced form credit models, and in particular the rating class model of Jarrow, Lando and Turnbull, we propose to model the firm value process as a time-changed Brownian motion. We are lead to consider modifying the classic first passage problem for stochastic processes to capitalize on this time change structure. We demonstrate that the distribution functions of such "first passage times of the second kind" are efficiently computable in a wide range of useful examples, and thus this notion of first passage can be used to define the time of default in generalized structural credit models. General formulas for credit derivatives are then proven, and shown to be easily computable. Finally, we show that by treating many firm value processes as dependent time changes of independent Brownian motions, one can obtain multifirm credit models with rich and plausible dynamics and enjoying the possibility of efficient valuation of portfolio credit derivatives.

Key words: Credit risk, structural credit model, time change, Lévy process, first passage time, default probability, credit derivative.

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1 Introduction

The structural approach to credit modelling, beginning with the works of Merton [Merton, 1974] and Black and Cox [Black & Cox, 1976], treats debt and equity as contingent claims (analogous to barrier options) on the firm's asset value process. While this unification of debt with equity is conceptually satisfying, the approach often leads to inconsistencies with intuition and observation, such as the zero short-spread property (a consequence of the predictable nature of the default time) and time inconsistency in Merton type models. Furthermore, it leads to technical difficulties when pushed to provide realistic correlations between different firms' defaults and with other market observables. Formulas in structural models tend to be either tractable but inflexible (when the firm value is taken to be geometric Brownian motion), or flexible but computationally intractable (when the firm value process is anything else).

Reduced-form (or "intensity-based") modelling, introduced by Jarrow and Turnbull [Jarrow & Turnbull, 1995], has been highly successful in providing remedies for these problematic aspects. It treats default as locally unpredictable, with an instantaneous hazard rate, but does away with the connection between default and the firm's asset value process.

Subsequent developments, such as the JLT model of Jarrow, Lando and Turnbull [Jarrow *et al.*, 1997] and its extensions [Lando, 1998, Arvanitis *et al.*, 1999], [Hurd & Kuznetsov, 2007], have to some extent bridged the gap between reduced form and structural models by positing a continuous time Markov chain to replace the firm value process as a determinant of credit quality, while retaining the concept of hazard rate in the form of dynamically varying Markov transition rates. The time of default is the first-hitting time of the default state, an absorbing state of the Markov chain. So-called hybrid models [Madan & Unal, 2000] (see also [Carr & Wu, 2005]) seek to tighten the connection with structural models by allowing the hazard rate to depend on the firm's equity value (stock price).

The purpose of the present paper is to propose and explore a particular mathematical structure, called a time-changed Brownian motion (TCBM), that can be used for consistent modelling of a firm's asset value process and its time of default as a first passage time. We aim to retain flexibility (to be able to match a wide range of possible credit spread curves), computational tractability (to permit efficient option valuation), and logical consistency with the paper of Black and Cox (by treating default as a first passage time for the firm value to hit a default threshold).

Many authors have used time-changed Brownian motions as models of log stock returns (a notable review paper is [Geman *et al.*, 2001]). When the time change is an independent Lévy process (Lévy subordinator), one obtains well known models such as the variance gamma (VG) model and the normal inverse Gaussian (NIG) model. Barndorff-Nielsen and Shephard [Barndorff-Nielsen & Shephard, 2001] have introduced time change models where the time change is an integrated mean-reverting jump process, while important stochastic volatility models such as Heston's model [Heston, 1993] arise from time changes that are integrated mean-reverting diffusions.

Following on the heels of these stock price models, it was natural to extend structural credit models by using time-changed geometric Brownian motions and other jump-diffusion processes to model the firm value (see [Zhou, 2001, Ruf & Scherer, 2006]). While this idea in principle cures some of the deficiencies of the classic Black-Cox model by adding flexibility and the possibility of unpredictable defaults, there is a huge price to pay in the difficulties of computing first passage distributions. Theoretical first passage results based on fluctuation theory [Bingham, 1975] and Wiener-Hopf factorization [Bertoin, 1996] are known, but exact formulas are rare. Kou and Wang [Kou & Wang, 2003] manage to solve the first passage problem for a specific class of jump-diffusion process, and Chen and Kou [Chen & Kou, 2005] use those results to extend the Black-Cox firm value model and the Leland-Toft model [Leland & Toft, 1996] for the optimal capital structure of the firm.

It was observed in [Hurd & Kuznetsov, 2007] that many computations in the JLT rating class framework, including evaluation of portfolio credit derivatives, are facilitated by treating credit migration processes as time-changed Markov chains. Since time-changed Markov chains can be viewed as discrete approximations to continuum-valued structural models, this last observation motivates our aim a careful analysis of first passage times for time-changed diffusions.

Therefore, in this paper, we propose to use time-changed Brownian motions to model the firm value process. To avoid the difficulties that arise in computing the associated first passage distribution and in analogy to the time-changed Markov chain models where the default state is an absorbing state, we are then lead to propose a specific variation of first passage time applicable to time-changed Brownian motions, but not to general jump diffusions. This variation, which we call the first passage time of the second kind, is designed to be decomposable by iterated conditional expectation, and thus can be computed much more efficiently in cases of interest. This concept is not new, having been used for example by Baxter [Baxter, 2006] in his computations of basket credit derivatives, but to our knowledge its modelling implications have not yet been fully explored.

Our purpose here is threefold. First we explore the mathematical structure of first passage times for time-changed Brownian motion, and provide a set of natural solvable examples that can be used in finance. By comparison of these examples with a range of existing stock price models, we thereby demonstrate the broad applicability of our framework to equity and credit modelling. Our second aim is to focus on structural models of credit where the firm value process is a general time-changed Brownian motion and the time of default is a first passage time of the second kind. We prove pricing formulas for defaultable zero coupon bonds and other credit derivatives, with and without stochastic recovery. This discussion demonstrates that time-changed Brownian motion can be the basis of single firm credit models consistent with the principles of no arbitrage, and with tractable valuation formulas for all important derivatives. Finally, we demonstrate how the single firm model can be extended to the joint default dynamics of many firms. Under a restrictive assumption on the correlation structure, analogous to the one-factor default correlation structure in copula models, we demonstrate the efficiency of valuation formulas for portfolio credit derivatives.

To avoid obscuring our most important results by focussing on a too-specific application, we postpone statistical work on the modelling framework to subsequent papers.

While we are hopeful that positive verification of the modelling assumptions on asset price datasets will ultimately show the viability of our framework, such a verification must proceed one application at a time, and would take us too far in the present paper. This will be the purpose of a series of future works, beginning with our paper [Hurd, 2007] on the joint modelling of credit and equity derivatives.

In outline, the paper proceeds as follows. Section 2 introduces the probabilistic setting and the definition and basic properties of TCBMs. The first passage problem for TCBMs is addressed in Section 3. Since the standard first passage problem for TCBMs exhibits no simplification over first passage problem for general jump-diffusions, we introduce an alternative notion, called the first passage time of the second kind, that capitalizes on the time change structure. It is this notion that is used in all subsequent developments. Sections 4 and 5 introduce the main categories of time changes, namely the Lévy subordinators and the integrated mean-reverting jump-diffusions. These two families are in a sense complementary, and together provide a rich and tractable family of TCBMs. Section 6 introduces the simplest structural credit models based on TCBMs, and runs through the valuation of some basic credit derivatives. Section 7 provides a brief numerical exploration of the single firm model. The multifirm extension is addressed in Section 8. We find that computational tractability strongly suggests that while the time change processes for different firms may (indeed should) be correlated, the underlying Brownian motions must be taken independent firm by firm.

2 Time-changed Brownian motion

Let $(\Omega, \mathcal{F}, P)$ be a probability space that supports a Brownian motion W and a non-decreasing process G with $G_0 = 0$, called the *time-change*. P may be thought of as either the physical or risk-neutral measure. Let $X_t = x + W_t + \beta t$ be the Brownian motion starting at x having constant drift β. We henceforth restrict the scope by assuming

Assumption 1. X and G are independent processes under P.

This assumption is mostly for simplicity: the more general case where X, G are dependent processes is of interest in finance. An important result proved by [Carr & Wu, 2005] allows us to extend out results to this case. A *time-changed Brownian motion (TCBM)* is defined to be a process of the form

$$L_t = X_{G_t}, \quad t \geq 0. \tag{2.1}$$

Identification of the components of such a TCBM leads to two subfiltrations of the natural filtration $\mathcal{F}_t$ (which we assume satisfies the "usual conditions"):

$$\begin{aligned} \mathcal{L}_t &= \sigma\{L_s : s \leq t\}, \\ \mathcal{G}_t &= \sigma\{G_s : s \leq t\}. \end{aligned} \tag{2.2}$$

We also consider the Brownian filtration $\mathcal{W}_t = \sigma\{W_s : s \leq t\}$.

In subsequent sections we give two important classes of examples of TCBMs, but for the remainder of this and the next section we consider the general case. We begin by defining *characteristic functions* Φ and *log-characteristic functions* $\Psi = \log \Phi$ for any $0 \le s \le t$ and $u \in \mathcal{D}$, $\mathcal{D}$ a domain in $\mathbb{C}$:

$$\begin{aligned}
\Phi^X_s(u,t) &= E[e^{iu(X_t - X_s)}|\mathcal{W}_s] = e^{i(\beta u + iu^2/2)(t-s)}, \\
\Phi^G_s(u,t) &= E[e^{iu(G_t - G_s)}|\mathcal{G}_s], \\
\Phi^L_s(u,t) &= E[e^{iu(L_t - L_s)}|\mathcal{L}_s].
\end{aligned} \tag{2.3}$$

These are to be understood as processes in the variable s. A simple calculation gives an essential formula

$$\begin{aligned}
\Phi^L_s(u,t) &= E[E[e^{iu(X_{G_t} - X_{G_s}}|\mathcal{F}_s \vee \mathcal{G}_t]|\mathcal{F}_s] \\
&= E[\Phi^X_{G_s}(u, G_t)|\mathcal{F}_s] = \Phi^G_s(\beta u - iu^2/2, t).
\end{aligned} \tag{2.4}$$

As we shall see, "solvable models" arise when Φ^G_s and hence Φ^L_s are explicit deterministic functions of an underlying set of Markovian variables. Characteristic functions and log-characteristic functions lead to formulas for moments $m^{(k)} = E[L^k_t]$ and cumulants $c^{(k)}$ for $k = 1, 2, \ldots$.

An important algebraic aspect of TCBM is their natural composition rules. If G, H are two independent time changes then $G + H$, $G \times H$ and $G \circ H$ are also TCBMs, and we have useful results such as $\Phi^{G+H}_s = \Phi^G_s \times \Phi^H_s$ and $\Phi^{G\circ H}_s = E[\Phi^G_{H_s}(u, H(t))|\mathcal{F}_s]$.

3 First passage distributions

In this section, we define two distinct notions of first passage time for a TCBM starting at a point $x \ge 0$ to hit zero.

Definition 1. For any TCBM $L_t = X_{G_t}$

1. The *first passage time of the first kind* is the $\mathcal{L}$-stopping time

$$t^{(1)} = \inf\{t | L_t \le 0\}. \tag{3.1}$$

   The corresponding stopped TCBM process is $L^{(1)}_t = L_{t\wedge t^{(1)}}$. Note that in general $L^{(1)}_{t^{(1)}} \le 0$, with the inequality possible at a time when G jumps.

2. The *first passage time of the second kind* is the $\mathcal{F}$-stopping time

$$t^{(2)} = \inf\{t | G_t \ge t^*\}, \tag{3.2}$$

   where $t^* = \inf\{t | X_t \le 0\}$. The corresponding stopped TCBM process is $L^{(2)}_t = X_{G_{t\wedge t^*}}$. Note that $L^{(2)}_{t^{(2)}} = 0$.

Remarks 2. 1. We view $t^{(2)}$ as an approximation of the usual first passage time $t^{(1)}$ with $t^{(1)} \geq t^{(2)}$. When G is a continuous process, the two definitions coincide. We can summarize the general situation by the phrase "the first passage time of the time-change of a process is greater than or equal to the time change of the first passage time of that process".

2. In general, $t^{(2)}$ is not an $\mathcal{L}$-stopping time. For more details, see [Geman *et al.*, 2001] who discuss the problem of inferring the time change G from observing the history of L.

3. When the time change is a pure jump process with unpredictable jumps, both stopping times are totally inaccessible. In general, they can be written as the minimum of a predictable stopping time and a totally inaccessible stopping time.

4. We can extend the second kind of first passage time to processes formed with composite time changes. For example, if G, H are independent time changes and $K_t = X_{(G\circ H)_t} = L_{H_t}$, we can define

$$t^{(3)} = \inf\{t : H_t \geq t^{(2)}\}, \;\; t^{(2)} = \inf\{t : G_t \geq t^*\}, \;\; t^* = \inf\{t | X_t \leq 0\},$$

and similarly higher order first passage times.

The precise distinction between $L^{(1)}$ and $L^{(2)}$ prior to the stopping time $t^{(i)}$ can be understood as follows. Conditioned on occurrence of a jump of size $\Delta G_t := G_{t^+} - G_{t^-} > 0$ at time t, and supposing $\beta = 0$ for simplicity, the distribution of $\Delta L^{(1)}_t$ is a Gaussian distribution of mean zero and variance ΔG_t, with a lower truncation point enforcing $\Delta L^{(1)}_t > -L^{(1)}_{t^-}$. On the other hand the conditional distribution of $\Delta L^{(2)}_t$ is the distribution of a Brownian motion at time ΔG_t conditioned to stay above $-L^{(2)}_{t^-}$ for $s \in [0, \Delta G_t]$. From this observation one can gain a clear qualitative picture of the differences between $L^{(1)}$ and $L^{(2)}$; in particular, one sees that the two distributions are almost identical except when $L^{(1)}_{t^-}/\sqrt{\Delta G_t} \gtrsim 2$.

Computation of the distributional properties of $t^{(1)}$ for a general TCBM is a difficult problem, with explicit solutions available only in a sparse set of examples. General properties have been established via fluctuation methods [Bingham, 1975] and Wiener-Hopf factorization [Bertoin, 1996]. For this reason, we instead focus our efforts on $t^{(2)}$. We begin by evaluating "structure" functions for drifting Brownian motion $L = X$ itself ($G_t = t$), for which the definitions of $t^{(1)}$ and $t^{(2)}$ coincide. The following well known formulas for Brownian motion are important for subsequent developments:

Proposition 1. *For any $x > 0$ let $L_t = X_t = x + W_t + \beta t$. Then*

1. *The cumulative distribution function $P(t, x, \beta) := P[t^{(1)} \leq t]$ for the first passage time of drifting Brownian motion is*

$$P(t, x, \beta) = N\left(\frac{-x - \beta t}{\sqrt{t}}\right) + e^{-2\beta x} N\left(\frac{-x + \beta t}{\sqrt{t}}\right). \tag{3.3}$$

For any $u \in \mathbb{C}$ with Im$(u) > -\beta^2/2$, the characteristic function of $t^{(1)}$, $\Phi(u, x, \beta) := E[e^{iut^{(1)}}]$, is

$$\Phi(u, x, \beta) = \int_0^\infty e^{iut} \left(\frac{\partial P(t, x, \beta)}{\partial t} \right) dt = \exp[-(\beta + \sqrt{\beta^2 - 2iu})x]. \tag{3.4}$$

2. *The conditional distribution function $P(l, t, x, \beta) := E[\mathbf{1}_{\{t^{(1)}>t\}}\mathbf{1}_{\{L_t \geq l\}}]$ is given by*

$$P(t, l, x, \beta) = N\left(\frac{x - l + \beta t}{\sqrt{t}}\right) - e^{-2\beta x} N\left(\frac{-x - l + \beta t}{\sqrt{t}}\right) \tag{3.5}$$

for $l \geq 0$ while the conditional characteristic function of L_t $\Phi(u, t, x, \beta) := E[\mathbf{1}_{\{t^{(1)}>t\}} e^{iuL_t}]$, is

$$\Phi(t, u, x, \beta) = e^{iu(x+\beta t) - u^2 t/2} P(t, 0, x, \beta + iu). \tag{3.6}$$

Here N is the complex analytic extension of the CDF of the standard normal random variable defined by the contour integral

$$N(z) = \int_{-z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx\ . \tag{3.7}$$

Proof: All but the last expression are standard results, so we sketch only the derivation of (3.6). From (3.5) we have

$$\begin{aligned}
\Phi(u, t, x, \beta) &= \int_0^\infty e^{iul} \left(-\frac{\partial P(l, t, x, \beta)}{\partial l} \right) dl \\
&= \int_0^\infty e^{iul} \frac{1}{\sqrt{2\pi t}} \left[e^{-(l-x-\beta t)^2/2t} - e^{-2\beta x - (l - \beta t + x)^2/2t} \right] dl.
\end{aligned}$$

One can compute these integrals by completing the square in each exponent and using the definition of N. Recombining the terms then leads to the result. □

These limited results reflect the difficulty in attaining insight into the first passage problem for general TCBMs, but show that Brownian motion itself is well understood. As we shall now see, the elegant properties of Brownian motion prove useful in the theory of the second kind of passage problem, for which the structure functions of $t^{(2)}$ are efficiently computable via an intermediate conditioning. Thus, for example:

$$\begin{aligned}
P^{(2)}(t, x) &:= P[t^{(2)} \leq t] = E[P[t^* \leq G_t | \mathcal{G}_\infty]] = \int_0^\infty P(y, x, \beta) \rho_t(y) dy, \\
P^{(2)}(l, t, x) &:= E[\mathbf{1}_{\{t^{(2)}>t\}} \mathbf{1}_{\{L_t \geq l\}}] \\
&= E[E[\mathbf{1}_{\{X_{G_t}>l\}} \mathbf{1}_{\{\min_{s \leq G_t} X_s > 0\}} | \mathcal{G}_\infty]] = \int_0^\infty P(l, y, x, \beta) \rho_t(y) dy, \qquad (3.8)
\end{aligned}$$

where ρ_t is the density of G_t and the functions P are given by (3.3) and (3.5). In cases of interest where the log characteristic function $\Psi^G(u,t)$ of the time change G_t is given in closed form, this formula can be made even more explicit with a modest amount of Fourier analysis.

Proposition 2. *For any $x > 0$ let $L_t = X_{G_t}$ be a TCBM.*

1. *For $t \geq 0$ suppose that $\bar\epsilon = \sup\{\epsilon \in \mathbb{R} : \Psi^G(-i\epsilon, t) < \infty\} > 0$. Then for any $\epsilon \in (0, \bar\epsilon)$, the cumulative distribution function for $t^{(2)}$, the first passage time of the second kind, is*

$$P^{(2)}(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\epsilon + iu} e^{-x[\beta+\sqrt{\beta^2+2(\epsilon+iu)}]} e^{\Psi^G(u-i\epsilon,t)} du. \tag{3.9}$$

2. *For any $l > 0$, the conditional probability density function of L_t is*

$$-\frac{\partial P^{(2)}(l,t,x)}{\partial l} = \int_{-\infty}^{\infty} e^{-iul}\left(e^{iux} - e^{-(2\beta+iu)x}\right) e^{\Psi^G(\beta u + iu^2/2, t)} du, \tag{3.10}$$

while for any $u \in \mathbb{R}$ the conditional characteristic function is

$$\Phi^{(2)}(u,t,x) = 2\pi\left(e^{iux} - e^{-(2\beta+iu)x}\right) e^{\Psi^G(\beta u + iu^2/2, t)}. \tag{3.11}$$

Proof: We prove (3.9) and (3.10) and leave the remaining formulas to the reader. For any $0 < \epsilon < \bar\epsilon$ and $u \in \mathbb{R}$,

$$e^{\Psi^G(u-i\epsilon,t)} = \int_0^\infty e^{iuy} e^{\epsilon y} \rho_t(y) dy$$

is an absolutely convergent integral. Hence by the Fourier Inversion Theorem,

$$\rho_t(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuy} e^{-\epsilon y} e^{\Psi^G(u-i\epsilon,t)} du, \quad y \geq 0.$$

By the Fubini Theorem, plugging this formula into (3.8) and reversing the order of integration leads to

$$P^{(2)}(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\Psi^G(u-i\epsilon,t)} \left[\int_0^\infty e^{-(\epsilon+iu)y} P(y,x,\beta) dy\right] du.$$

Finally, integration by parts in y and the use of (3.4) leads to formula (3.9).

By the dominated convergence theorem applied to the l derivative of (3.5),

$$-\frac{\partial P^{(2)}(l,t,x)}{\partial l} = \int_0^\infty \frac{1}{\sqrt{2\pi y}}\left[e^{-(l-\beta y-x)^2/2y} - e^{-2\beta x-(l-\beta y+x)^2/2y}\right] \rho_t(y) dy.$$

The standard Gaussian integral

$$\frac{1}{\sqrt{2\pi y}} e^{-a^2/2y} = \int_{-\infty}^{\infty} e^{-iua-u^2y/2} du$$

allows us to expand the integrand above, and use of the Fubini theorem to interchange the order of integration leads to

$$-\frac{\partial P^{(2)}(l,t,x)}{\partial l}=\int_{-\infty}^{\infty}e^{-iul}\left[e^{iux}-e^{-(2\beta+iu)x}\right]\left[\int_0^{\infty}e^{i(\beta u+iu^2/2)y}\rho_t(y)dy\right]du.$$

The inner integral equals $e^{\Psi^G(\beta u+iu^2/2,t)}$, leading to the desired formula (3.10). □

We pause here to give some typical formulas for structure functions of composite time change processes. Let G, H be independent time changes leading to processes $L_t = X_{G_t}$, $K_t = L_{H_t} = X_{(G\circ H)_t}$, $\tilde{K}_t = X_{H_t+G_t}$. Then the default probability function for $t^{(2)} = \inf\{s|H_s + G_s > t^*\}$ can be efficiently computed by

$$P^{(2)}(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\epsilon+iu}e^{-x[\beta+\sqrt{\beta^2+2(\epsilon+iu)}]}e^{\Psi^G(u-i\epsilon,t)+\Psi^H(u-i\epsilon,t)}du. \tag{3.12}$$

Other structure functions can be managed in a similar way.

4 Lévy subordinated Brownian motions

The first important class of TCBMs arises by taking G to be a Lévy time change, in other words a *Lévy subordinator*. Lévy processes are the general class of continuous time stochastic processes with stationary and independent increments. In addition to their interest in the theory of stochastic processes, they have found important uses in mathematical finance, where they are used as models for log-stock price processes. Much of the analysis connected with a Lévy process L_t is based on its *characteristic triple* $(\tilde{b},\tilde{c},\rho)_h$, in terms of which its log characteristic function takes the form

$$\Psi^L(u,t) := \log E[e^{iuL_t}] = t\left[i\tilde{b}u-\tilde{c}^2u^2/2+\int_{\mathbb{R}\setminus 0}[e^{iuy}-1-iuyh(y)]\rho(y)dy\right]. \tag{4.1}$$

Here ρ is a measure on $\mathbb{R}\setminus 0$. For ease of exposition in what follows, we set the truncation function $h(y)$ to zero, which is permissible by adopting the restrictive condition that $|x|\wedge 1$ should be ρ-integrable. Our main results extend to the general case where $|x|^2\wedge 1$ is ρ-integrable. See [Cont & Tankov, 2004, Cherny & Shiryaev, 2002] for general discussions of Lévy processes.

The following result is given as Exercise 3.33 in [Cherny & Shiryaev, 2002], and identifies the type of process that can be expressed as a *Lévy-subordinated Brownian motion* (LSBM) $L_t := X_{G_t}$:

Theorem 3. *Supposing $L_0 = x$, the following are equivalent statements:*

1. *L is a Lévy process with characteristic triple $(\tilde{b},\tilde{c},\rho)_0$ where $\tilde{b},\tilde{c}\ge 0$. The density $\rho(y)$ is nowhere zero on $\mathbb{R}$ and can be written in the form*

$$\rho(y)=\int_0^{\infty}\frac{1}{\sqrt{2\pi z}}e^{-(y-\beta z)^2/z}\nu(z)dz \tag{4.2}$$

for $\beta = \tilde{b}/\tilde{c}$ and some measure ν on $(0,\infty)$. (In case $\tilde{c} = 0$, then $\tilde{b}$ must be zero, and $\rho(\sqrt{z})$ must be a completely monotone function on $(0,\infty)$.)

2. *$L_t := X_{G_t}$ for Brownian motion X with drift $\beta \in \mathbb{R}$ and G a Lévy subordinator with characteristic triple $(b, 0, \nu)_0, b \geq 0$, and ν a measure on $(0,\infty)$.*

Here are some examples of such processes that have been used as models of logarithmic stock returns:

1. The exponential model with parameters (a, b, c) arises by taking G to be the increasing process with drift $b \geq 0$ and jump measure $\nu(z) = ce^{-az}, c, a > 0$ on $(0,\infty)$. The log characteristic function of G_t is

$$\Psi^G(u,t) := \log E[e^{iuG_t}] = t[ibu + iuc/(a - iu)].$$

The resulting time-changed process $L_t := X_{G_t}$ has triple $(\beta b, b, \rho)_0$ with

$$\rho(y) = \frac{c}{\sqrt{\beta^2 + 2a}} e^{-(\sqrt{\beta^2+2a}+\beta)(y)^+ - (\sqrt{\beta^2+2a}-\beta)(y)^-},$$

where $(y)^+ = \max(0, y), (y)^- = (-y)^+$. This forms a four dimensional subclass of the six-dimensional family of exponential jump diffusions applied to finance in [Kou & Wang, 2003].

2. The VG model [Madan & Seneta, 1990] arises by taking G to be a gamma process with drift defined by the characteristic triple $(b, 0, \nu)_0$ with $b \geq 0$ (usually b is taken to be 0) and jump measure $\nu(z) = ce^{-az}/z, c, a > 0$ on $(0,\infty)$. The log characteristic function of $G_t, t = 1$ is

$$\Psi^G(u,t) := \log E[e^{iuG_t}] = t[ibu - c\log(1 - iu/a)].$$

The resulting time-changed process has triple $(\beta b, b, \rho)_0$ with

$$\rho(y) = \frac{c}{|y|} e^{-(\sqrt{\beta^2+2a}+\beta)(y)^+ - (\sqrt{\beta^2+2a}-\beta)(y)^-}.$$

3. The normal inverse Gaussian model (NIG) with parameters $\tilde{\beta}, \tilde{\gamma}$ [Barndorff-Nielsen, 1997] arises when G_t is the first passage time for a Brownian motion with drift $\tilde{\beta} > 0$ to exceed the level γt. Then

$$\Psi^G(u,t) = -\tilde{\gamma}t(\tilde{\beta} + \sqrt{\tilde{\beta}^2 - 2iu})$$

and the resulting time-changed process has log-characteristic function

$$\Psi^L(u,t) = ix\mu - t\tilde{\gamma}[\tilde{\beta} + \sqrt{\tilde{\beta}^2 + u^2 - 2i\tilde{\beta}u}].$$

In these and certain other financially relevant Lévy models, the log-characteristic function of G is explicit, leading as we will see shortly to efficient numerical computations that involve intensive use of the Fast Fourier Transform. The log-characteristic function, viewed as the cumulant generating function, also facilitates calibration. We can identify the moments $m^{(k)} := EL_t^k, k = 1, 2, \ldots$ of a LSBM L_t, or more conveniently, its cumulants $c^{(k)}$:

$$
\begin{aligned}
c^{(1)} &:= m^{(1)} = t(\beta b + \beta m_\nu^{(1)}), \\
c^{(2)} &:= m^{(2)} - (m^{(1)})^2 = t(b + m_\nu^{(1)} + \beta^2 m_\nu^{(2)}), \\
c^{(3)} &:= m^{(3)} - 3m^{(2)}(m^{(1)})^2 + 2(m^{(1)})^3 \\
&= t(3\beta m_\nu^{(2)} + \beta^3 m_\nu^{(3)}), \\
c^{(4)} &:= m^{(4)} - 4m^{(3)}m^{(1)} + 3(m^{(2)})^2 + 12m^{(2)}(m^{(1)})^2 - 12(m^{(1)})^4 \\
&= t(3m_\nu^{(2)} + 6\beta^2 m_\nu^{(3)} + \beta^4 m_\nu^{(4)}).
\end{aligned}
$$

Here, $m_\nu^{(k)}$ denotes the kth moment of the Lévy measure of the time change G_t.

5 Affine TCBMs

For our second important class of time changes, G_t has differentiable paths, and the corresponding TCBMs are diffusions (processes with continuous paths) which exhibit "stochastic volatility". We focus here on a class we call ATCBMs ("affine" TCBMs), for which G is taken in the class of positive mean-reverting CIR-jump processes introduced by [Duffie & Singleton, 1999]:

$$
\begin{aligned}
G_t &= G_t^{(1)} + G_t^{(2)} = \int_0^t (\lambda_s^{(1)} + \lambda_s^{(2)}) ds, \\
d\lambda_t^{(1)} &= (a - b\lambda^{(1)})dt + \sqrt{2c\lambda_t^{(1)}} dW_t^{(1)}, a, b, c > 0, \\
d\lambda_t^{(2)} &= -\tilde{b}\lambda^{(2)} dt + dJ_t.
\end{aligned} \tag{5.1}
$$

Here J is taken identical to the exponential Lévy subordinator with parameters $(\tilde{a}, 0, \tilde{c})$ defined in example 1 of the previous section.

The essential computations for characteristic functions

$$\Phi^{(i)}(u, t; \lambda) := E[e^{iuG_t^{(i)}} | \lambda_0^{(i)} = \lambda], i = 1, 2$$

of such affine time changes are described in many papers. The following formulas are proved in the appendix of [Hurd & Kuznetsov, 2007]:

Proposition 4. *The characteristic functions $\Phi^{(i)} := \Phi^{G^{(i)}}, i = 1, 2$, both have the exponential affine form*

$$\Phi^{(i)}(u, t; \lambda) = e^{-\phi^{(i)}(u,t) - \lambda\psi^{(i)}(u,t)}. \tag{5.2}$$

The functions $\phi^{(i)}$ and $\psi^{(i)}$ are explicit:

1.

$$
\text{(5.3)} \qquad \begin{cases} \psi^{(1)}(u,t) = -\psi_2 + \left(1 + \frac{c}{\gamma}\psi_1\left(e^{\gamma t} - 1\right)\right)^{-1}\psi_2, \\ \phi^{(1)}(u,t) = -a\psi_1 t + \frac{a}{c}\log\left(1 + \frac{c}{\gamma}\psi_1\left(e^{\gamma t} - 1\right)\right), \end{cases}
$$

with constants ψ_1, ψ_2 and γ given by

$$
\text{(5.4)} \qquad \begin{cases} \gamma = \sqrt{b^2 - 4iuc}, \\ \psi_1 = \frac{b+\gamma}{2c}, \\ \psi_2 = \frac{b-\gamma}{2c}. \end{cases}
$$

2.

$$
\text{(5.5)} \qquad \begin{cases} \psi^{(2)}(u,t) = \left(\frac{-iu}{\tilde{b}}\right)e^{-\tilde{b}t} + \frac{-iu}{\tilde{b}}, \\ \phi^{(2)}(u,t) = \tilde{c}t - \frac{\tilde{a}\tilde{c}}{\tilde{a}\tilde{b}-iu}\log\left(\frac{(\tilde{a}\tilde{b}-iu)e^{\tilde{b}t}+iu}{\tilde{a}\tilde{b}}\right). \end{cases}
$$

Combining this result with the results of Section 3 leads to closed-form, or "close-to-closed-form" solutions for the structure functions of the associated first-passage time $t^{(2)}$.

The ATCBM model with time change $G^{(1)}$ is equivalent to the Heston stochastic volatility model for stock returns [Heston, 1993], with zero correlation (hence zero leverage effect). Stock price models with time change $G^{(2)}$, and extensions thereof, were introduced in [Barndorff-Nielsen & Shephard, 2001]. It is also worth remarking that the class of LSBM processes can be achieved using the limits of $G^{(2)}$-type time changes as the mean-reversion rate is taken to infinity. Thus it is clear from the examples of the past two sections that the class of TCBM models with time change written as a sum of these three types is rich enough to describe a wide range of asset classes in finance. We now focus on the application to structural models of credit risk.

6 Structural Credit Models

The credit modelling paradigm of Black and Cox [Black & Cox, 1976] assumes that default of a firm is triggered as the debtholders exercise a "safety covenant" when the value of the firm falls to a specified level. It makes sense therefore to assume that the time of default is the time of first passage of the firm value process V_t below a specified lower threshold function $K(t)$. In our setup, we assume

Assumptions 1. 1. There is a vector $Z_t = [\tilde{r}_t, \lambda_t^{(1)}, \lambda_t^{(2)}]$ of independent processes with $\lambda^{(i)}$ chosen as in Section 5. $\tilde{r}$ is a CIR process with characteristic function $\Phi^{\tilde{r}}(u,t)$ given in the form (5.3).

2. The process $L_t = \log(V_t/K(t)) = X_{G_t}, X_t = x + W_t + \beta t$, called the "log-leverage ratio", is a TCBM. The time change is given by

$$G_t = bt + G_t^{(1)} + G_t^{(2)} + G_t^{(3)}, \quad b \geq 0. \tag{6.1}$$

Here $G_t^{(i)} = \int_0^t \lambda_s^{(i)} ds, i = 1, 2$ are defined as in Section 5 with characteristic functions $\Phi^{(i)}(u, t; \lambda^{(i)})$ while $G^{(3)}$ is a Lévy subordinator $G_t^{(3)}$ with characteristics $(0, 0, \nu)_0$ and characteristic function $\Phi^{(3)}(u, t)$.

3. The time of default is $t^{(2)}$, the first passage time of the second kind.

4. The spot interest rate is $r_t = \tilde{r}_t + m_1 \lambda_t^{(1)} + m_2 \lambda_t^{(2)}$ for non-negative coefficients m_1, m_2.

5. A constant recovery fraction $R < 1$ under the recovery of treasury mechanism is paid on defaultable bonds at the time of default. (This is for simplicity only: as in [Hurd & Kuznetsov, 2007] we can allow R_t to be a general affine process.)

Note that the usual structural approach for jump diffusions is based on the first passage time of the first kind, and leads to technical difficulties: our innovation is to consider instead the second kind of first passage time. The following proposition gives formulas for default probabilities and default-free and defaultable zero coupon bond prices.

Proposition 5. *Let the initial credit state of the firm be specified by initial values $L_0 = x$ and $Z_0 = [\tilde{r}_0, \lambda_0^{(1)}, \lambda_0^{(2)}]$.*

1. *The probability that default occurs before $t > 0$ is given by*

$$\begin{aligned} P[t^{(2)} \leq t] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(\epsilon+iu)bt}}{\epsilon + iu} e^{-x[\beta + \sqrt{\beta^2 + 2(\epsilon+iu)}]} \\ &\quad \times \Phi^{(1)}(u - i\epsilon, t; \lambda_0^{(1)}) \Phi^{(2)}(u - i\epsilon, t; \lambda_0^{(2)}) \Phi^{(3)}(u - i\epsilon, t) du. \end{aligned} \tag{6.2}$$

2. *The time 0 price $B(T)$ of default-free zero coupon bond with maturity T is*

$$B(T) = \Phi^{\tilde{r}}(i, T; \tilde{r}_0) \Phi^{(1)}(im_1, T; \lambda_0^{(1)}) \Phi^{(2)}(im_2, T; \lambda_0^{(2)}). \tag{6.3}$$

3. *The time 0 price $\bar{B}(T)$ of defaultable zero coupon bond with maturity T, under constant fractional recovery of treasury is given by*

$$\begin{aligned} \bar{B}(T) &= B(T) + (R-1) \frac{\Phi^{\tilde{r}}(i, T; \tilde{r}_0)}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(\epsilon+iu)bT}}{\epsilon + iu} e^{-x[\beta + \sqrt{\beta^2 + 2(\epsilon+iu)}]} \\ &\quad \times \Phi^{(1)}(u - i(\epsilon - m_1), T; \lambda_0^{(1)}) \Phi^{(2)}(u - i(\epsilon - m_2), T; \lambda_0^{(2)}) \Phi^{(3)}(u - i\epsilon, T) du. \end{aligned} \tag{6.4}$$

The above pricing formulas are explicit functions of the initial values L_0, Z_0; as time develops, prices are deterministic functions of the processes L_t and Z_t. We adopt the point of view that Z contains information about the drivers of general credit markets, while L reflects firm specific information.

7 Numerical results

The structural credit modelling framework of the previous section is designed with a great deal of flexibility in mind, and it would take us far afield to seek a fully calibrated model. Instead, in this section we strip out much of the complexity, and merely exhibit a simple set of model parametrizations that generate plausible credit spread curves and derivative prices, thereby demonstrating the computational efficiency.

In Figure 1, we compare the thirty year zero recovery yield spread and default probability density curves under the four parametrizations of the exponential jump model shown in Table 1, for a pure geometric Brownian motion (Model A) and three pure jump processes. All versions of the model are specified so that L_t has fixed annualized variance 0.09 (i.e. the firm value has $\sigma = 30\%$ volatility) and mean log rate of return $-\sigma^2/2$. We observe that the yield spreads equalize as maturity increases, but show the completely different short time behaviour expected from the presence of jumps.

Figure 2 shows the thirty year zero recovery yield spreads in Model B for four firms which differ in their initial distance-to-default values $L_0 = 0.3, 0.6, 1.0, 2.0$.

	Model A	Model B	Model C	Model D
L_0	1.5	1.5	1.5	1.5
a	1	11.59	111.6	1111.6
b	0.09	0	0	0
c	0	1	10	100
β	-0.5	-0.5	-0.5	-0.5
σ^2	0.09	0.09	0.09	0.09

Table 1: Parameter values for the exponentially subordinated Brownian motion model.

8 Structural Models for Many Firms

An outstanding difficulty in credit risk is finding a modelling framework that extends naturally and efficiently to a large number of firms, while allowing for a rich default dependence structure. The present setup of time-changed Brownian motions turns out to be just such a framework. Consider M firms, where for each $j = 1, 2, \ldots, M$, the jth firm is governed by its firm value process V_t^j, default trigger threshold $K^j(t)$ and log-leverage ratio process

$$
\begin{aligned}
L_t^j &= \log V_t^j / K^j(t) = X^j_{\tilde{H}_t^j}, \\
X_t^j &= x^j + W_t^j + \beta^j t. \qquad (8.1)
\end{aligned}
$$

Here, for the jth firm we take parameters $x^j, \beta^j \geq 0$, and a possibly firm dependent time change $\tilde{H}_t^j$.

Assumptions 2. The joint dynamics of multifirm defaults is determined by the first passage to zero of the log-leverage ratio processes L_t^j. The time change processes $\tilde{H}^j$ are given jointly in the form

$$\tilde{H}_t^j = b^j t + \alpha^j G_t + H_t^j \tag{8.2}$$

with $b^j, \alpha^j \geq 0$ and time changes G, H^j having the form given by (6.1). Finally, we assume that $G, X^1, \ldots, X^M, H^1, \ldots, H^M$ are mutually independent processes.

In models of this type, the maximal correlation structure is obtained by setting the firm-specific time changes H_t^j to zero. However, since the underlying Brownian motions W^j are independent, maximal correlation does not mean the defaults are fully correlated. This setting can be interpreted as a generalized Bernoulli mixing model, in the sense of [Bluhm *et al.*, 2003] and [McNeil *et al.*, 2005], where the mixing random variable is G_t. That is, the default states of all firms at time t are conditionally independent Bernoulli random variables, conditioned on the value of G_t. Define the conditional probability that $t^j \leq t$ conditioned on $G_t = y$:

$$P^j(x^j, t, y) := E[1_{\{t^j \leq t\}} | G_t = y] = \int_0^\infty F(x^j, \beta^j, b^j t + \alpha^j y + z) d\rho_t^j(z) \tag{8.3}$$

where ρ_t^j is the PDF of H_t^j. The following formula extends (3.9), and is proved exactly the same way: for any $0 < \epsilon < \bar{\epsilon}^j$ and $y \in \mathbb{R}$

$$P^j(x, t, y) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{1}{\epsilon + iu} e^{(\epsilon+iu)(b^j t + \alpha^j y)} e^{-x[\beta^j + \sqrt{(\beta^j)^2 + 2(\epsilon+iu)}]} e^{t\Psi_{H^j}(u - i\epsilon)} du. \tag{8.4}$$

Now, for any subset $\sigma \subset \{1, 2, \ldots, M\}$, the unconditional probability that the firms in default at time t are precisely the firms in σ is given by

$$\begin{aligned} P[t^j \leq t, j \in \sigma; t^j > t, j \notin \sigma] &= \\ &\int_0^\infty \prod_{j \in \sigma} P^j(x^j, t, y) \prod_{j \notin \sigma} (1 - P^j(x^j, t, y)) \rho_t(y), \end{aligned} \tag{8.5}$$

where ρ_t is the distribution function of G_t.

There are by now well-known techniques that under the assumption of conditionally dependent defaults, reduce the computation of CDO tranches to intensive computation of the conditional default probabilities $P^j(x^j, t, y)$. [Hurd & Zhang, 2007] explores the promising use of these techniques for computing CDO pricing in our multi-firm dynamic credit framework.

9 Conclusions

We have studied the first passage problem for a class of jump diffusions that are important for financial modelling, namely the Lévy subordinated Brownian motions. It was seen that the first passage time of the second kind presents some key advantages over the classic definition of first passage time, particularly computational tractability and the extension to multi-dimensional processes.

Based on these good properties, we defined a pure first passage structural model of default, and obtained computable formulas for the basic credit instruments, namely bonds and CDSs. The resultant formulas resolve a fundamental deficiency of the classic Black-Cox formula, namely the zero short spread property, and provides needed flexibility to match details of yield spreads.

Finally, we outlined an extension to many firms in which dependence stems from systemic components to the time change, while the underlying Brownian motions are independent and firm specific. The resulting multifirm framework has many of the advantageous properties that have been observed in related work of [Hurd & Kuznetsov, 2006], in particular, a conditional independence structure that enables semianalytic computations of large scale basket portfolio products such as CDOs. A detailed investigation of this model's uses in portfolio credit VaR and CDO pricing is the subject of future work.

References

[Arvanitis *et al.*, 1999] Arvanitis, A., Gregory, J., & Laurent, J.-P. 1999. Building models for credit spreads. *Journal of derivatives*, **6**, 27–43.

[Barndorff-Nielsen & Shephard, 2001] Barndorff-Nielsen, O., & Shephard, N. 2001. Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics. *J. royal statist. soc., series b*, **63**.

[Barndorff-Nielsen, 1997] Barndorff-Nielsen, O. E. 1997. Normal inverse Gaussian distribution and stochastic volatility modelling. *Scandinavian journal of statistics*, **24**, 1–13.

[Baxter, 2006] Baxter, M. 2006. *Dynamic modelling of single-name credits and CDO tranches*. working paper.

[Bertoin, 1996] Bertoin, J. 1996. *Lévy processes*. Cambridge: Cambridge University Press.

[Bingham, 1975] Bingham, N. H. 1975. Fluctuation theorems in continuous time. *Adv. appl. prob.*, **7**, 705–766.

[Black & Cox, 1976] Black, F., & Cox, J. C. 1976. Valuing corporate securities. *J. finance*, **31**, 351–367.

[Bluhm *et al.*, 2003] Bluhm, C., Overbeck, L., & Wagner, C. 2003. *An introduction to credit risk modelling*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL.

[Carr & Wu, 2005] Carr, P., & Wu, L. 2005. *Stock options and credit default swaps: A joint framework for valuation and estimation*. working paper.

[Chen & Kou, 2005] Chen, N., & Kou, S. 2005. *Credit spreads, optimal capital structure, and implied volatility with endogenous default and jump risk*. Working paper available at `http://www.defaultrisk.com/pp_price_71.htm`.

[Cherny & Shiryaev, 2002] Cherny, A. S., & Shiryaev, A. N. 2002. *Change of time and measures for Lévy processes*. Lectures for the Summer School "From Lévy Processes to Semimartingales: Recent Theoretical Developments and Applications to Finance", Aarhus 2002.

[Cont & Tankov, 2004] Cont, R., & Tankov, P. 2004. *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL.

[Duffie & Singleton, 1999] Duffie, D., & Singleton, K. 1999. Modeling term structures of defaultable bonds. *Review of financial studies*, **12**, 687–720.

[Geman *et al.*, 2001] Geman, H., Madan, D. B., & Yor, M. 2001. Time changes for Lévy processes. *Math. finance*, **11**(1), 79–96.

[Heston, 1993] Heston, S. L. 1993. A closed–form solution for options with stochastic volatility with applications to bond and currency options. *Rev. financial studies*, **6**(2), 327–343.

[Hurd, 2007] Hurd, T. R. 2007. *Joint modelling of equity and credit derivatives via time changed Brownian motions*. working paper.

[Hurd & Kuznetsov, 2006] Hurd, T. R., & Kuznetsov, A. 2006. *Fast CDO computations in the Affine Markov Chain Model*. Preprint available at `http://www.defaultrisk.com/pp_crdrv_65.htm`.

[Hurd & Kuznetsov, 2007] Hurd, T. R., & Kuznetsov, A. 2007. Affine Markov chain models of multifirm credit migration. *J. of credit risk*, **3**, 3–29.

[Hurd & Zhang, 2007] Hurd, T. R., & Zhang, X. 2007. *Dynamic CDO modelling with time changed Brownian motions*. Working Paper.

[Jarrow *et al.*, 1997] Jarrow, R., Lando, D., & Turnbull, S. 1997. A Markov model for the term structure of credit risk spreads. *Review of financial studies*, 481–523.

[Jarrow & Turnbull, 1995] Jarrow, R. A., & Turnbull, S. M. 1995. Pricing derivatives on financial securities subject to credit risk. *Journal of finance*, **50**, 53–85.

[Kou & Wang, 2003] Kou, S. G., & Wang, H. 2003. First passage times of a jump diffusion process. *Adv. in appl. probab.*, **35**(2), 504–531.

[Lando, 1998] Lando, D. 1998. On Cox processes and risky bonds. *Review of derivatives research*, **2**, 99–120.

[Leland & Toft, 1996] Leland, H. E., & Toft, K. B. 1996. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *J. finan.*, **51**, 987–1019.

[Madan & Seneta, 1990] Madan, D., & Seneta, E. 1990. The VG model for share market returns. *Journal of business*, **63**, 511–524.

[Madan & Unal, 2000] Madan, D., & Unal, H. 2000. A two-factor hazard-rate model for pricing risky debt and the term structure of credit spreads. *Journa mert74 l of financial and quantitative analysis*, **35**(1), 43–65.

[McNeil *et al.*, 2005] McNeil, A. J., Frey, R., & Embrechts, P. 2005. *Quantitative risk management: Concepts, techniques, and tools*. Princeton University Press.

[Merton, 1974] Merton, R. C. 1974. On the pricing of corporate debt: the risk structure of interest rates. *J. finance*, **29**, 449–470.

[Ruf & Scherer, 2006] Ruf, J., & Scherer, M. 2006. *Pricing corporate bonds in an arbitrary jump-diffusion model based on an improved Brownian-bridge algorithm*. working paper downloadable at `www.stat.columbia.edu/~ruf/WP_Ruf_Scherer_2006.pdf`.

[Zhou, 2001] Zhou, C. 2001. The term structure of credit spreads with jump risk. *J. bus. and finan.*, **25**, 2015–2040.

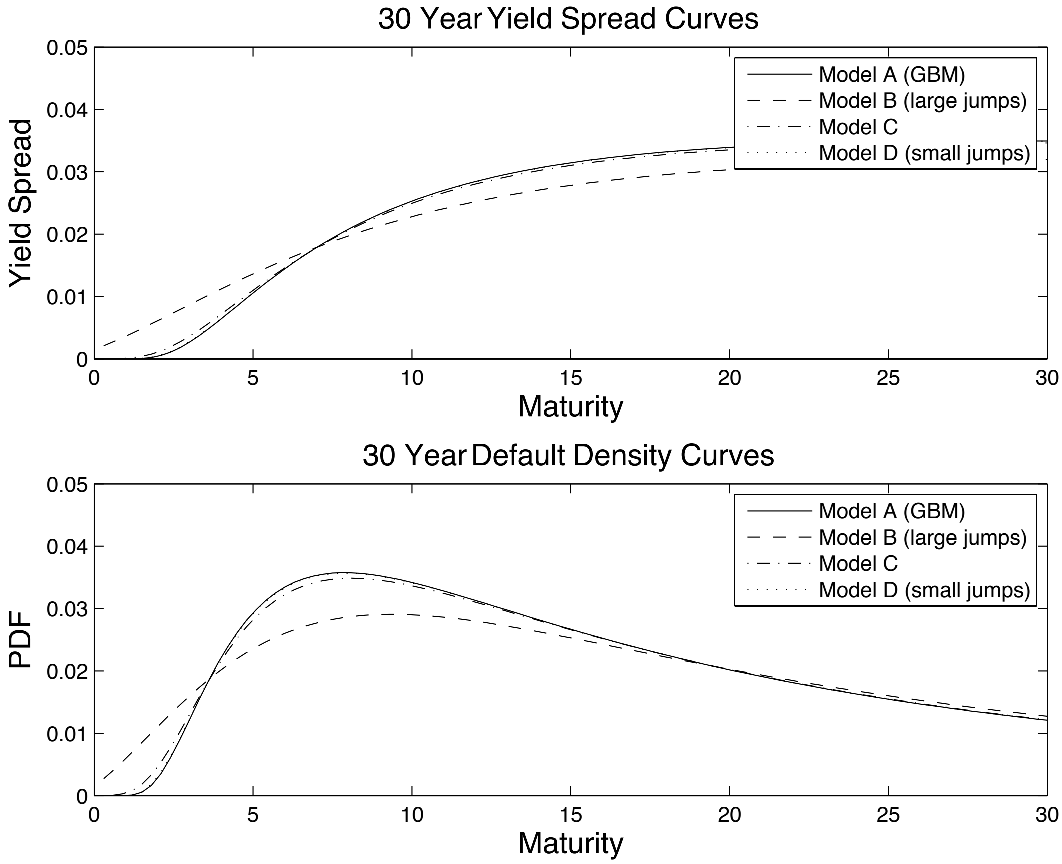

Figure 1: Thirty year yield spread and default PDF curves for 4 versions of the exponentially subordinated Brownian motion credit risk model.

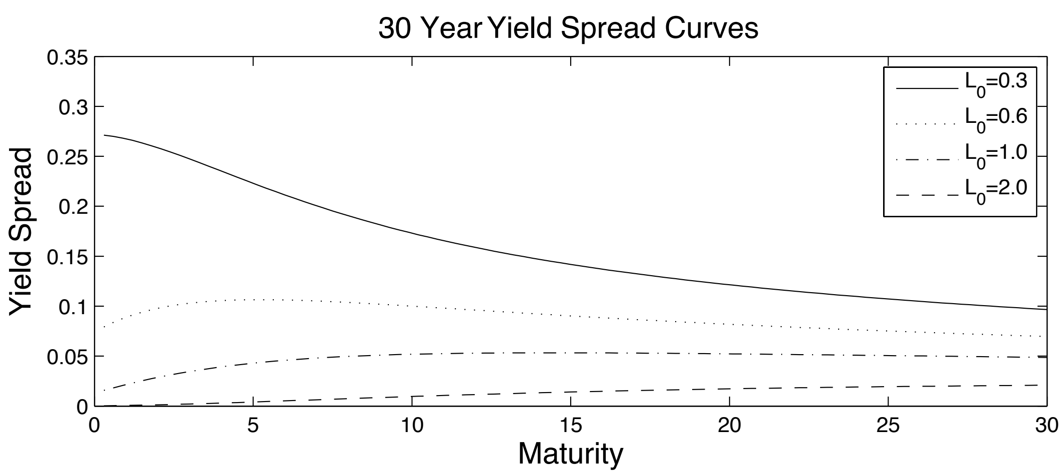

Figure 2: Thirty year yield spread for Model B with four different values $L_0 = 0.3, 0.6, 1.0, 2.0$.