RESEARCH STATEMENT

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Part 1. Nonlinear Equations of Monge Ampère type

Introduction

During my post-doctoral fellowship term at McMaster I worked in conjunction with Eric T. Sawyer on problems related to the Monge Ampère equation,

$$\det\left(D^2 f(x)\right) = k(x) \geq 0, \qquad x \in \Omega \subset \mathbb{R}^n, \tag{1}$$

where $D^2 f = \left(\frac{\partial^2 f}{\partial_i \partial_j}\right)_{1\leq i,j\leq n}$ and Ω is a uniformly convex domain. The uniformly elliptic case ($0 < c \leq k < \infty$) has been extensively studied and a rather complete theory of a priori regularity of solutions as well as existence and regularity of solutions of the Dirichlet boundary problem

$$\begin{aligned} \det\left(D^2 f(x)\right) &= k(x), \qquad x \in \Omega \\ f(x) &= \varphi(x), \qquad x \in \partial\Omega \end{aligned} \tag{2}$$

is available [1], [2], [3], [4]. More precisely, in [4] the authors prove that the Monge Ampère equation is subelliptic, that is, weak solutions to (2) with k, φ and $\partial\Omega$ smooth (C^∞), verify an a priori estimate of the form

$$\|f\|_{C^{2,\alpha}\left(\bar{\Omega}\right)} \leq C(\Omega, \varphi, k) < \infty, \tag{3}$$

for some $\alpha > 0$ independent of f. An immediate consequence of (3) and the classic Schauder estimates is that the solution f is smooth.

In the degenerate case, when k is allowed to vanish, simple examples show that we can not expect solutions to the Dirichlet problem (2) to be smooth even if φ, $\partial\Omega$ are smooth, and k vanishes only at one point. In [5], Guan, Trudinger and Wang proved that if k is merely nonnegative and smooth, then there always exists a unique convex generalized solution to (2), which lies in $C^{1,1}\left(\overline{\Omega}\right)$. An explicit example of Siboni (see [6]) show that $C^{1,1}$ cannot be improved under this level of generality.

Still the question of conditions which would enforce solutions of (2) to have a higher order of smoothness remained open in the degenerate case, even though $C^{1,1}$ regularity is the best possible result for a general nonnegative right hand side, one would like to know under which extra conditions for $k \geq 0$, if any, a unique smooth solution to (2) exists. In [6] P. Guan proved that if k vanishes to a finite order, together with some other conditions, then the Monge Ampère operator is subelliptic. The first positive result in which k was allowed to vanish to an infinite order, is a two dimensional result and is due to Sawyer and Wheeden [7]. Their approach uses the partial Legendre transform

$$\begin{aligned} s &= x, \\ t &= f_x, \end{aligned} \tag{4}$$

to write equation (1) in two dimensions as a quasilinear equation in divergence form

$$\partial_s^2 w + \partial_t \left(k\left(s, w\right) \partial_t w \right) = 0, \tag{5}$$

where $w\left(s,t\right) = y$. Then the authors prove a priori estimates for smooth solutions of (5), under the conditions

$$k\left(x, y\right) > 0, \qquad x \neq 0, \tag{6}$$

$$\left|\partial_y k\left(x, y\right)\right| \leq B\left(k\left(x, y\right)\right)^{\frac{3}{2}}, \qquad \text{for some } B > 0. \tag{7}$$

The growth condition (7) says that as k vanishes it becomes a function of the x variable only, that is, as k degenerates, equation (5) becomes essentially more linear. The result in [7] establishes the first known conditions for the Monge Ampère equation to be *hypoelliptic*, that is, smooth solutions to (1) have interior C^∞ semi-norms controlled by constants depending only on k, $\left\|f\right\|_\infty$, the constant B and the distance to $\partial\Omega$. This result is sharp in the sense that there exist smooth k verifying (6) for which the corresponding Monge Ampère equation is not subelliptic, that is, an estimate like (3) does not hold. In [7] the authors also established a priori estimates and regularity of solutions for the two dimensional degenerate Gaussian curvature problem. In the next two sections, some aspects of a joint ongoing project with E.T. Sawyer and R.L. Wheeden are discussed. Preprints for this work are available at the writer's website.

An n-dimensional quasilinear equation

Motivated by the powerful results in [7], we considered the n-dimensional case for equation (5).

$$\partial_1^2 w + \sum_{i=2}^{n} \partial_i \left(k_i\left(x, w\right) \right) \partial_i w = 0, \tag{8}$$

where $x \in \Omega \subset \mathbb{R}^n$, w is a smooth function, and k is smooth and nonnegative. This is a step on the direction to treat the n-dimensional degenerate Monge Ampère equation and related geometric problems, like the Gaussian curvature problem, scalar curvature problem, etc. The relation of (8) with (1) is discussed in the next section. Under the nonnegativity conditions

$$k_i > 0, \qquad x_i \neq 0,$$

motivated by the two dimensional case (6), we proved that smooth solutions to (1) have C^∞ semi-norms controlled by constants depending only on k, $\left\|w\right\|_\infty$, $\left\|\nabla w\right\|_\infty$ and the distance to $\partial\Omega$. This result is the n-dimensional generalization of the two dimensional version in [7], also proved without the assumption (7). If k furthermore verifies conditions similar in character to (7), then we proved that the first derivatives of f can be controlled in the interior by constants depending only on k, $\left\|f\right\|_\infty$, B and the distance to $\partial\Omega$. Even though this theorem generalizes results in [7], it does not immediately apply to the n-dimensional Monge Ampère equation, since equation (8) does not arise as a higher dimensional partial Legendre transform of (1).

The general partial Legendre transform

The partial Legendre transform (4), translates the *fully nonlinear* problem (1) into the *quasilinear* equation (5). Whereas the degeneracy and smoothness of the coefficients are preserved, the quasilinear character has apparent advantages over the fully nonlinear one. This prompted the quest for an n-dimensional version of this useful transformation, that would allow the use of quasilinear theory for the treatment of (1). A useful transformation of this sort was not available until now. We where able to write the Monge Ampère equation in n dimensions as an $(n-1)$-dimensional system of quasilinear equations with smooth coefficients. This system has very special properties, stemming from the geometric character of (1). This properties allow us to treat the $(n-1)$-dimensional system and derive properties for the solutions to (1) in several different cases, not to be discussed in detail here. This stage is presently being developed and we have some full and some partial results as well as conjectures on future applications of the techniques developed.

Given f and k smooth verifying (1), we consider the transformation

$$\begin{aligned} s &= x_1, \\ t_i &= f_i, \qquad i = 2, \dots, n, \end{aligned}$$

and the complementary variables

$$\begin{aligned} u &= f_1, \\ v_i &= x_i, \qquad i = 2, \dots, n. \end{aligned}$$

We notice that the following "Cauchy Riemann" equations hold

$$\begin{aligned} \frac{\partial u}{\partial s} &= k \det \left(\frac{\partial \vec{v}}{\partial \vec{t}} \right) \\ \frac{\partial u}{\partial \vec{t}} &= - \left(\frac{\partial \vec{v}}{\partial s} \right)^{tr}. \end{aligned}$$

From which it follows the quasilinear system

$$\frac{\partial^2 v_j}{\partial s^2} + \frac{\partial}{\partial t_j} \left(k \det \left(\frac{\partial \vec{v}}{\partial \vec{t}} \right) \right) = 0, \qquad j = 2, \dots, n, \tag{9}$$

which can be rewritten as

$$\operatorname*{div}_{(s,\vec{t})} A \nabla_{(s,\vec{t})} v_j = 0, \qquad j = 2, \dots, n,$$

where A is a $n \times n$ symmetric matrix. The remarkable fact about this system is that it can also be written as a nondivergence quasilinear system of the same order. This is due to the special character of the matrix A stemming from the geometry inherent to (1). We exploit this special feature to adapt techniques of [6], [7] and others to obtain a collection of results. In particular, the first n-dimensional subellipticity result for solutions to degenerate Monge Ampère equations is obtained in some special cases. This is work currently in progress, up to date preprints can be found at the writer's website.

Part 2. **Elliptic Nondivergence Linear Equations**

Regularity of the Dirichlet Problem with L^p data

On my doctoral research at the University of Minnesota, I studied the Dirichlet problem

$$\begin{aligned} \mathcal{L}u &= 0 \quad \text{in } D \\ u &= f \quad \text{on } \partial\Omega \end{aligned} \tag{10}$$

where Ω is a bounded, Lipschitz domain in $\mathbb{R}^n$, $\mathcal{L}u = a^{i,j}(x)\,\partial_{i,j}u(x)$ (the repeated indices summation convention is used), $\partial_{i,j} = \frac{\partial^2\cdot}{\partial x_i \partial x_j}$ and the matrix $A(x) = (a^{i,j}(x))_{i,j=1}^n$ is real, symmetric, uniformly bounded and verifies the ellipticity condition

$$0 < \lambda\,|\xi|^2 \leq a^{i,j}(x)\,\xi_i\,\xi_j \quad \text{for all } x \in r,\ \xi \in \mathbb{R}^n \setminus \{0\}\,. \tag{11}$$

In [8], the authors constructed examples of operators $\mathcal{L}$ as above, with continuous coefficients, and for which solutions to (10) are not unique when the boundary data lies in $L^p(\partial\Omega)$, for any p, $1 \leq p \leq \infty$. This means that the harmonic measure induced by $\mathcal{L}$ on $\partial\Omega$, is *singular* with respect to *surface measure* σ (c.f. [9]). This arises the question of under which conditions on the coefficients A, we can assure uniqueness of solutions of (10), with proper control depending only on the L^p norm of the boundary data, and the geometry of the domain. This question is related to the existence of a "good definition" of solutions to $\mathcal{L}u = 0$ when the coefficients of $\mathcal{L}$ are not smooth, and it was formulated as an open problem in C. Kenig's book [9] (problem 3.3.9., see also 3.3.5–3.3.8).

In [10], we obtained *sufficient* conditions on the coefficients a^{ij} that assure the induced harmonic measure to be absolutely continuous with respect to surface measure on $\partial\Omega$. This results where inspired on the existent theory for divergence form operators, specially on the remarkable results in [11], where the conditions imposed are also shown to be sharp.

There are a number of interesting open problems to be considered in this theory, even in the case of continuous coefficients. Refer the writer's website for available papers and preprints as well as a more detailed description of future projects in this area.

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