THE DIRICHLET PROBLEM FOR INFINITELY DEGENERATE QUASILINEAR EQUATIONS
(IN PREPARATION)

CRISTIAN RIOS, ERIC T. SAWYER, AND RICHARD L. WHEEDEN

Abstract. We prove *a priori* bounds for solutions of a class of quasilinear equations of the form

$$\operatorname{div} \mathbf{A}(x,w)\nabla w = \vec{g}(x,w)\cdot\nabla w + f(x,w),$$

where $x=(x_1,\dots,x_n)$, and where f, $\vec{g}=\left(g^i\right)_{1\le i\le n}$ and $\mathbf{A}=\left(a_{ij}\right)_{1\le i,j\le n}$ are smooth. The square symmetric matrix $\mathbf{A}$ is permitted to vanish to infinite order as a quadratic form. We show that if w is a continuous weak solution of the Dirichlet problem in a domain Ω with smooth data, then $w\in C^\infty(\Omega)$.

1. Introduction

It is well-known that if $\mathbf{A}$ is elliptic, and $\mathbf{A}$ and $\mathbf{b}$ are smooth functions of their arguments, then quasilinear operators in the divergence form

$$\mathcal{L}w = \operatorname{div}\mathbf{A}(x,w,\nabla w) + \mathbf{b}(x,w,\nabla w)$$

are hypoelliptic: any weak solution w to $\mathcal{L}w=0$ is smooth (see [3]). When $\mathcal{L}$ is *subelliptic* - i.e. ellipticity fails only to finite order - then hypoellipticity still holds if $\mathcal{L}$ is *linear* (see e.g. Treves [9]). When $\mathcal{L}$ is linear but fails to be subelliptic, the situation is more delicate. For example, in [2], Fediĭ showed that the two dimensional operator

$$\partial_x^2 + k(x)\,\partial_y^2 \tag{1.1}$$

is hypoelliptic if k is smooth and positive for all $x\neq 0$. In this case k is allowed to vanish at any rate at $x=0$. However, $\partial_x^2 + k(x)\,\partial_y^2 + \partial_z^2$ is hypoelliptic in $\mathbb{R}^3$ only for certain orders of vanishing (see [4]). A quasilinear version of operators of the form (1.1) arises when one considers two dimensional Monge-Ampère equations

$$u_{ss}u_{tt} - u_{st}^2 = k(s,t,u,u_s,u_t), \qquad (s,t)\in\widetilde{\Omega}\subset\mathbb{R}^2, \tag{1.2}$$

together with the classical partial Legendre transformation $(x,y)=T(s,t)$ given by

$$\left\{\begin{array}{lcl} x & = & s \\ y & = & u_t \end{array}\right. . \tag{1.3}$$

Indeed, assuming that T is invertible, (1.2) and (1.3) lead to the two dimensional quasilinear equation

$$\partial_x^2 w + \partial_y\left\{k\left(x, w(x,y), r(x,y), v(x,y), y\right)\partial_y w\right\} = 0, \qquad (x,y)\in\Omega = T\left(\widetilde{\Omega}\right), \tag{1.4}$$

verified in the weak sense by $w(x,y)=t$, where $r(x,y)=u(s,t)$ and $v(x,y)=u_s(s,t)$. In [8], two of the authors extended Fediĭ's two dimensional regularity result for linear

---

Date: November 28, 2003.

equations to certain solutions w obtained through the transformation (1.3) from a solution of (1.2). The coefficient k considered in [8] is assumed to satisfy

$$|k_t(s,t,z,p,q)| \leq C\, k(s,t,z,p,q)^{\frac{3}{2}}, \qquad (s,t,z,p,q) \in \Omega \times \mathbb{R}^3, \tag{1.5}$$

that is, k is required to become more independent of the second variable as it degenerates. Notice that the coefficient k in (1.1) is independent of the second variable, and then (1.5) is automatically true. The result in [8] establishes that degenerate two dimensional Monge-Ampère equations (1.2) with smooth right-hand side k satisfying (1.5) are hypoelliptic. This was the first known hypoellipticity result for infinitely degenerate Monge-Ampère equations. More general equations than (1.4) are also treated in [8], including the equation for prescribed Gaussian curvature.

In this work we consider an n-dimensional quasilinear operator of the form

$$\mathcal{L}w(x) = \operatorname{div} \mathbf{A}(x, w(x)) \nabla w(x), \qquad x \in \Omega, \tag{1.6}$$

where ∇ denotes the gradient operator with respect to the x variable, the matrix $\mathbf{A}$ is symmetric and satisfies

$$\sum_{i=1}^{n} k^i(x,z)\,\xi_i^2 \leq \xi^t \mathbf{A}(x,z)\,\xi \leq \Lambda \sum_{i=1}^{n} k^i(x,z)\,\xi_i^2, \quad \text{for all } \xi \in \mathbb{R}^n, \tag{1.7}$$

for some $\Lambda \geq 1$, the functions $k^i(x,z)$ are smooth and nonnegative in a domain $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$, and Ω is the projection of Γ onto $\mathbb{R}^n$. We denote by $\vec{k}$ the vector $\left(k^1, \ldots, k^n\right)$.

Remark 1.1. *We assume that $\vec{k}$ has continuous second order derivatives in Γ and satisfies the following:*

- *There exists $C \geq 1$ such that for $(x,z) \in \Gamma$,*

$$0 \leq k^i(x,z) \leq C, \qquad i = 1, \ldots, n. \tag{1.8}$$

- *There exists $c > 0$ such that for each $(x,z) \in \Gamma$,*

$$\max_{1 \leq i \leq n} k^i(x,z) \geq c > 0. \tag{1.9}$$

- *For every $x \in \Omega$ and $\varepsilon > 0$, there exist $0 < R_1, \ldots, R_n \leq \varepsilon$ such that if $\mathcal{R}$ denotes the box centered at x defined by*

$$\mathcal{R} = [x_1 - R_1, x_1 + R_1] \times \cdots \times [x_n - R_n, x_n + R_n],$$

then

$$k^i(y,z) > 0 \text{ whenever } y \in T_i = \partial\mathcal{R} \setminus \{y : |y_i - x_i| = R_i\}, \quad (y,z) \in \Gamma, \tag{1.10}$$

i.e., $k^i(y,z)$ is positive if $(y,z) \in \Gamma$ and y lies on any face of the box $\mathcal{R}$ whose normal vector is perpendicular to the y_i-axis.

Note that if $\vec{k}$ satisfies (1.8), $k^1 \equiv 1$ and $k^i(x,z)$ is positive for $x \in \Omega$ away from the x_i-axis, i.e.,

$$k^i(x,z) > 0 \quad \text{if} \quad x_i' \equiv (x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_n) \neq \mathbf{0}, \quad (x,z) \in \Gamma, \tag{1.11}$$

for $2 \leq i \leq n$, then (1.9) and (1.10) are trivially verified.

In practice, since our theorems are local, we assume without loss of generality that

$$k^1(x,z) = 1, \qquad \text{for all } (x,z) \in \Gamma, \tag{1.12}$$

which clearly implies (1.9).

Definition 1.2. *Given smooth real functions* $\vec{g} = \left(g^1, g^2, \ldots, g^n\right)$ *and* f*, defined on* $\Omega \times \mathbb{R}$*, we say that* $w(x)$ *is a weak solution of*

$$\mathcal{L}w = \operatorname{div} \mathbf{A}(x,w) \nabla w = \vec{g}(x,w) \cdot \nabla w + f(x,w), \qquad x \in \Omega \tag{1.13}$$

in Ω *if* $\nabla w \in L^1_{\text{loc}}(\Omega)$ *and* w *belongs to the weighted Sobolev space*

$$H^1_{\text{loc}}(\Omega, \mathbf{A}) = \left\{ w : \int \varphi \;\; (\nabla w)^t \mathbf{A}(x,w) \nabla w \, dx < \infty, \quad \textit{for all } \varphi \geq 0 \in C^\infty_c(\Omega) \right\},$$

and if for any $\varphi \in C^\infty_c(\Omega)$*, we have*

$$\int (\nabla \varphi)^t \mathbf{A}(x,w) \nabla w \, dx = -\int \left[\varphi \, \vec{g}(x,w) \cdot \nabla w + \varphi \, f(x,w)\right] dx.$$

Definition 1.3. *Note that for* $w \in H^1_{\text{loc}}(\Omega, \mathbf{A})$ *we have* $(\nabla w)^t \mathbf{A}(x,w) \nabla w \in L^1_{\text{loc}}(\Omega)$*, and since* $\mathbf{A}$ *is a nonnegative matrix, the vector function*

$$\nabla_k w = \sqrt{\mathbf{A}(x,w)} \nabla w \tag{1.14}$$

is well-defined and $\nabla_k w \in L^2_{\text{loc}}(\Omega)$*. We call* $\nabla_k w$ *the* k-gradient *of* w *in* Ω*. Sometimes we will need to apply the derivation (1.14) to functions other than* w*; in such a case we denote*

$$\nabla_{k,w} \varphi = \sqrt{\mathbf{A}(x,w)} \nabla \varphi. \tag{1.15}$$

The subindex k in the definition of $\nabla_k w$ refers to the relation (1.7) between $\mathbf{A}$ and the vector $\vec{k}$. This is an abuse of notation since the correspondence between $\mathbf{A}$ and $\vec{k}$ is not unique ($\vec{k}$ is a smooth vector *equivalent* in size to the eigenvalues of $\mathbf{A}$), but we adopt the notation for convenience.

Our main result, Theorem 1.8 below, states that under certain hypotheses on the coefficients $\mathbf{A}$, every *continuous* weak solution w of (1.13) is infinitely differentiable, and all of its derivatives are locally controlled by $\|w\|_\infty$. The conditions imposed on the coefficients allow them to vanish to infinite order, so the quasilinear operator $\mathcal{L}$ is not in general uniformly elliptic or even subelliptic. This is the first known hypoellipticity result for infinitely degenerate quasilinear equations in n dimensions.

In our first theorem we establish *a priori* local control of all the derivatives of a smooth solution w of (1.13) in Ω, the control being in terms of both w and ∇w. This theorem, and its main application in this work, Theorem 1.8, include the results in [8] used to obtain regularity for Monge-Ampère equation in two dimensions.

Before stating our *a priori* estimate, it will be convenient to recall the classical inequality

$$|\nabla_{x,z} k(x,z)| \leq B\sqrt{k(x,z)}, \qquad (x,z) \in L, \tag{1.16}$$

for a compact subset L of Γ, and its more general form,

$$|\nabla_{x,z} k(x,z)| \leq C \left\{ \left\|\nabla^2_{x,z} k\right\|^{\frac{1}{2}}_\infty + \left(dist\left((x,z), \partial\Gamma\right)\right)^{-\frac{1}{2}} \right\} \sqrt{k(x,z)}, \qquad (x,z) \in \Gamma, \tag{1.17}$$

if k is merely nonnegative with bounded second derivatives on a domain Γ (see e.g. the appendix in [8]). Inequality (1.16) is crucial in our calculations, and although it has an analogue for diagonal matrices, it does not extend to general matrix functions. Consequently, we ask our coefficient matrix $\mathbf{A}$ to satisfy

$$|\partial_x \mathbf{A}(x,z)\,\xi|^2 + |\partial_z \mathbf{A}(x,z)\,\xi|^2 \leq B^2 \xi^t \mathbf{A} \xi, \quad \text{for all } \xi \in \mathbb{R}^n, \quad (x,z) \in L. \tag{1.18}$$

Note that condition (1.18) is equivalent to $(\partial_x \mathbf{A}(x,z))^2 + (\partial_z \mathbf{A}(x,z))^2 \leq B^2\,\mathbf{A}$ in the sense of bilinear forms, i.e. for all $\xi \in \mathbb{R}^n$,

$$\xi^t \left[(\partial_x \mathbf{A}(x,z))^2 + (\partial_z \mathbf{A}(x,z))^2 \right] \xi = |\partial_x \mathbf{A}(x,z)\,\xi|^2 + |\partial_z \mathbf{A}(x,z)\,\xi|^2 \leq B^2 \xi^t \mathbf{A} \xi.$$

Assuming that (1.18) holds,it follows from (1.14) in Definition 1.3 that

$$\|\varphi\,(\partial_x \mathbf{A}(x,v))\,\nabla v\|_{L^2}^2 + \|\varphi\,(\partial_z \mathbf{A}(x,v))\,\nabla v\|_{L^2}^2 \leq B^2\,\|\varphi \nabla_k v\|_{L^2}^2 \tag{1.19}$$

for any $v \in H^1_{\text{loc}}(\Omega, \mathbf{A})$ (see Definition 1.2) and any smooth cutoff function φ supported in Ω. For convenience, we assume $B \geq 1$ henceforth. For the nonhomogeneous term arising from $\vec{g}$, we assume that

$$|\vec{g}(x,z) \cdot \xi|^2 \leq B^2\,\xi^t \mathbf{A} \xi. \tag{1.20}$$

Condition (1.20) says that $B^{-1}\vec{g}$ defines a *subunit* vector field with respect to $\mathbf{A}$ in the sense that

$$\left| B^{-1}\vec{g} \cdot \nabla v \right|^2 \leq (\nabla v)^t\,\mathbf{A} \nabla v = |\nabla_k v|^2\,. \tag{1.21}$$

We will also need some further terminology. Let $\mathcal{P}_c(\Gamma)$ denote the collection of all compact subsets of Γ. We will say that a real-valued function f defined on $\mathcal{P}_c(\Gamma)$ is increasing if $f(L_1) \leq f(L_2)$ whenever $L_1, L_2 \in \mathcal{P}_c(\Gamma)$ with $L_1 \subset L_2$. We denote by Ω an open bounded connected subset of $\mathbb{R}^n$, and we assume that the graph of the solutions considered is contained in an open bounded connected domain $\Gamma \subset \mathbb{R}^{n+1}$ such that

$$\Omega \times \{0\} \subset \Gamma \subset \Omega \times \mathbb{R},$$

that is, a solution w satisfies $(x, w(x)) \in \Gamma$ for all $x \in \Omega$.

Theorem 1.4. *Suppose that $\vec{k}(x,z)$ is C^2 in a domain Γ as above, $\mathbf{A}$, $f(x,z)$, $\vec{g}(x,z)$ are C^∞ in Γ, where $\mathbf{A}$ satisfies (1.18), $\vec{k}(x,z)$ satisfies Hypothesis 1.1, and $\vec{g}$ satisfies (1.20). Let ζ, $\varkappa$ be smooth cutoff functions supported in Ω, with $\varkappa = 1$ on the support of ζ. Then, for every multi-index α, there is a real-valued function $\mathcal{C}_\alpha(\sigma, L)$, defined for $(\sigma, L) \in [0,\infty) \times \mathcal{P}_c(\Gamma)$ and increasing in each variable separately, so that for all smooth solutions w of (1.13) in Ω with $(x, w(x)) \in L$ for all x in the support of ζ,*

$$\|\zeta D^\alpha w\|_\infty \leq \mathcal{C}_\alpha\left(\|\varkappa \nabla w\|_\infty, L\right). \tag{1.22}$$

The constant $\mathcal{C}_\alpha(\sigma, L)$ depends on Γ, ζ, $\varkappa$, B and $\vec{k}$, but is independent of $\vec{g}$ and f.

Note that the right-hand side of (1.22) includes an implicit bound on w through the restriction that $(x, w(x)) \in L$ when $\zeta(x) \neq 0$.

Remark 1.5. *The special case of Theorem 1.4 that is proved in* [8] *($n = 2$, $\vec{g} = \mathbf{0}$, $f = 0$, $\mathbf{A}$ diagonal and $\vec{k}(x_1, x_2, z) = (1, k(x_1, x_2, z))$ independent of x_2) requires, for its main application to Monge-Ampère equations, an explicit control of the constants $\mathcal{C}_\alpha$ in terms of the coefficients $\vec{k}(x,z)$. More precisely, it is necessary to obtain uniform estimates for solutions of a family of operators of the form*

$$\mathcal{L}_\varepsilon = \partial_x^2 + \partial_y\left(k(x, w(x,y)) + \gamma\right)\partial_y, \qquad 0 \leq \gamma \leq 1.$$

The same explicit control is available in our case. Indeed, $\mathcal{C}_\alpha$ depends on the ratio M_i/δ_i, where

$$\delta_i = \min\left\{ k^i(x,z)\,,\; x \in T_i,\; (x,z) \in \Gamma \right\},$$

$$M_i = \left\| \nabla_x k^i \right\|_{L^\infty(\Gamma)}\,,$$

with T_i as in (1.10) for an appropriate fixed R_i which depends on α and $\vec{k}$. This fact is embedded in the proof of Theorem 1.4, although we omit the routine but tedious details of its proof for the sake of clarity.

Corollary 1.6. *Let $\mathcal{D}$ be the divergence form operator*

$$\mathcal{D}w=\partial_1^2 w+\sum_{i=2}^{n}\partial_i a^i(x,w)\,\partial_i w,$$

where a^i are smooth and nonnegative in Γ. Suppose that $f(x,z)$, $\vec{g}(x,z)$ are C^∞ in Γ as above, and there is a positive constant B such that $\frac{1}{B}\vec{g}$ is subunit with respect to $\vec{a}$, i.e.

$$\left|\vec{g}(x,z)\cdot\xi\right|^2\le B^2\sum_{i=1}^{n}a^i(x,z)\,\xi_i^2.$$

Let ζ, $\varkappa$ be smooth cutoff functions supported in Ω, with $\varkappa=1$ on the support of ζ. Then, for every multi-index α, there is a real-valued function $\mathcal{C}_\alpha(\sigma,L)$, defined for $(\sigma,L)\in[0,\infty)\times\mathcal{P}_c(\Gamma)$ and increasing in each variable separately, so that for all smooth solutions w of

$$\mathcal{D}w=\vec{g}(x,w)\cdot\nabla w+f(x,w),$$

in Ω with $(x,w(x))\in L$ for all x in the support of ζ,

$$\left\|\zeta D^\alpha w\right\|_\infty\le\mathcal{C}_\alpha\left(\left\|\varkappa\nabla w\right\|_\infty,L\right).$$

The constant $\mathcal{C}_\alpha(\sigma,L)$ depends on Γ, ζ, $\varkappa$, B and $\vec{k}$, but is independent of $\vec{g}$ and f.

Our main application of these a priori estimates is the following hypoellipticity result for (infinitely degenerate) quasilinear equations of the form (1.13). Because of the local character of the theorem, we replace condition (1.9) by (1.12) i.e., we assume that $k^1\approx 1$. As in the special two dimensional case contained in [8], we assume extra conditions on the coefficients, namely, we require that the *nonlinear* and the *infinitely degenerate* characters do not occur simultaneously in the following sense.

Remark 1.7. *In the next theorem, we assume in addition that $\mathbf{A}$ and $\vec{g}$ satisfy*

$$\left|\partial_z\mathbf{A}(x,z)\,\xi\right|^2\le CB^2\left(k^*(x,z)\right)^2\sum_{i=1}^{n}k^i(x,z)\,\xi_i^2,\tag{1.23}$$

with $k^*(x,z)=\min_{i=1,\dots,n}\left\{k^i(x,z)\right\}$,

$$\left|\partial_z\left(\partial\mathbf{A}(x,z)\right)\xi\right|^2+\left|\partial_z^2\mathbf{A}(x,z)\,\xi\right|^2\le CB^2\sum_{i=1}^{n}k^i(x,z)\,\xi_i^2,\tag{1.24}$$

$$\left|\partial_z g^i(x,z)\right|^2\le B^2k^*(x,z)\,k^i(x,z),\qquad i=1,\dots,n,\tag{1.25}$$

and

$$\left|\partial_z\left(\partial g^i(x,z)\right)\right|^2+\left|\partial_z^2 g^i(x,z)\right|^2\le B^2k^i(x,z),\qquad i=1,\dots,n,\tag{1.26}$$

where ∂ denotes a generic first order partial derivative $\partial_{x_1},\dots\partial_{x_n}$.

The extra vanishing condition (1.23) on $\mathbf{A}_z$ is a stronger form of the part of (1.18) involving $\mathbf{A}_z$. While in the scalar case inequality (1.18) always holds for any C^2 nonnegative $k(x,z)$, and it takes the form (1.16), the more restrictive version (1.23) takes the form

$$\left|\partial_z k(x,z)\right|\le B\left(k(x,z)\right)^{\frac{3}{2}},\tag{1.27}$$

(see also (1.5)) in the scalar case and it does not hold in general for nonnegative $k(x,z)$.

Theorem 1.8. *Suppose that* $\mathbf{A}$, $f(x,z)$, $\vec{g}(x,z)$, $\vec{k}(x,z)$, Γ *are as in Theorem 1.4, and they also satisfy Hypothesis 1.7. Then, if* w *is a* weak solution *of (1.13) in* Ω, *which is* continuous *in* Ω, *then* $w \in C^{\infty}(\Omega)$ *and for every multi-index* α,

$$\|\zeta D^{\alpha} w\|_{\infty} \le \mathcal{C}_{\alpha}(L). \tag{1.28}$$

Note that in the case $n=2$, $\mathbf{A}$ diagonal, (i.e. $\mathbf{A}(x,z) = \operatorname{diag}(1, a(x_1,x_2,z)))$, $f=0$ and $\vec{g}=\mathbf{0}$, Hypothesis 1.7 reduces to $a(x,z)$ satisfying (1.27) and $a(x_1,\cdot,z) \not\equiv 0$ in the variable x_2, as x_2 ranges through any open interval containing the origin. Indeed, by (1.7) we can take $k^2(x,z) = a(x,z)$, and since in this case $k^* = k^2$, and if $\mathbf{A}$ satisfies. Hypotheses 1.1 and 1.7, then from (1.23) we have that $a(x,z)$ satisfies (1.27). Moreover, from (1.10) we have that for every $\varepsilon > 0$, there exists $0 < R_1$, $R_2 < \varepsilon$ such that $a(x_1, R_2, z) > 0$ for all $-R_1 \le x_1 \le R_1$.

As an example, we consider diagonal matrices of the form $\mathbf{A} = \operatorname{diag}\left(1, k^2, \ldots, k^n\right)$ i.e.

$$\mathbf{A}(x,z) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & k^2(x,z) & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & k^n(x,z) \end{pmatrix},$$

where k^i are smooth nonnegative functions satisfying Hypothesis 1.1, and such that for some integer $N \ge 1$,

$$C^{-1}\left|k^j(x,z)\right|^N \le k^i(x,z) \le C\left|k^j(x,z)\right|^{-N}, \qquad 1 < i,j \le n, \quad (x,z) \in \Gamma, \tag{1.29}$$

and for $i = 1, \ldots, n$,

$$\left|k_z^i\right| \le B^*\left(k^i\right)^{N+\frac{1}{2}}, \qquad i = 1, \ldots, n,$$

with N as in (1.29). The $\mathbf{A}$ satisfies the hypotheses of Theorem 1.8. In particular, if $k(x,z)$ is nonnegative and it satisfies $|k_z| \le B k^{\frac{3}{2}}$, then $\mathbf{A} = \operatorname{diag}(1,k,\ldots,k)$ is an *admissible* matrix for Theorem 1.8.

As a striking consequence of Theorem 1.8 in $\mathbb{R}^2$, if $k(x,y,z)$ is smooth, nonnegative, satisfies

$$|\partial_z k| \le B^*(k)^{\frac{3}{2}}, \tag{1.30}$$

and $k(\cdot,\cdot,0)$ *does not identically vanish on any horizontal line segment* in Ω, then any continuous weak solution w to

$$\mathcal{L}w = \partial_x^2 w + \partial_y k(x,y,w(x,y))\,\partial_y w, \qquad (x,y) \in \Omega, \tag{1.31}$$

is smooth in Ω. Indeed, since in this case $k^1 \equiv 1$ and (1.24) is trivially satisfied with $N=1$. It only rests to check that k satisfies (1.10). Since $k(\cdot,\cdot,0)$ does not vanish on any horizontal segment, given $\varepsilon > 0$ and $(x,y) \in \Omega$, there exist $\bar{x} < x < \bar{x}'$ with $|x-\bar{x}| = |x-\bar{x}'| = R_1 < \varepsilon$, such that $k(\bar{x},y,0) > 0$ and $k(\bar{x}',y,0) > 0$. From (1.30) it follows that $k(\bar{x},y,z) > 0$ and $k(\bar{x}',y,z') > 0$ whenever $(\bar{x},y,z)$, $(\bar{x}',y,z') \in \Gamma$ (see Lemma 5.1 on the appendix), (1.10) follows then from the continuity of k with respect to the second variable.

In [8], a special case of Theorem 1.8 was obtained in two dimensions, and it was used to treat Monge-Ampère equations. The techniques used in this work closely follow those in [8]. The new elements introduced here to establish the n-dimensional results of theorems 1.4 and 1.8 are contained in the proof of Lemma 2.12, and the *a priori* estimates for the gradient, Lemma 5.3.

1.1. **Notation.** Throughout this paper we will use C to denote a constant that may change from line to line, but that is independent of any significant quantities. In general C may depend on the dimension n, $\vec{k}$ and the fixed cutoff functions defined below. We will use calligraphic $\mathcal{C}$ to denote a function of one or more variables, increasing in each variable separately, that may also change from line to line, but that remains independent of any significant quantities apart from its variables. We will only denote the dependence of $\mathcal{C}$ with respect to variables others than the dimension n, $\vec{k}$ and the cutoff functions selected.

Given two operators S and T we denote by $[S,T]$ the commutator operator

$$[S,T]=ST-TS.$$

We will abbreviate partial derivatives using subindexes, and superindexes will be reserved to indicate components of vector fields. In this way, $\vec{k}=\left(k^1,\ldots,k^n\right)$, and if we denote $\partial_z=\partial_{n+1}$, we write

$$\begin{aligned} k_j^i &= \partial_j k^i\left(x,z\right), \qquad k_{jl}^i=\partial_l\partial_j k^i\left(x,z\right), \\ k_z^i &= \partial_z k^i\left(x,z\right), \qquad k_{jz}^i=\partial_z\partial_j k^i\left(x,z\right), \qquad \ldots\text{etc.} \end{aligned} \tag{1.32}$$

Given two nonnegative cutoff functions ζ, $\xi\in C_0^\infty\left(\Omega\right)$, we say that ξ *supports* ζ and denote $\xi\succeq\zeta$ if

$$\xi=1 \qquad \text{on a neighborhood of support}\left(\zeta\right).$$

Note in particular that if $\xi\succeq\zeta$ then $\zeta\,\xi=\zeta$ and $\left\|\zeta\right\|_\infty\xi\geq\zeta$. Here and henceforth $\left\|f\right\|_\infty$ denotes the supremum norm of f and $\left\|f\right\|_{L^p(\Omega)}$ the L^p-norm of f in the set Ω,

$$\left\|f\right\|_{L^p(\Omega)}=\left(\int_\Omega\left|f\right|^p dx\right)^{\frac{1}{p}},$$

when $\Omega=\mathbb{R}^n$ we omit mentioning the set. We also adopt the notation

$$\left\|f\right\|_0=\left\|f\right\|_{L^2}.$$

We will use nonnegative cutoff functions adapted to our operator as in [2], and defined as follows:

Definition 1.9. *Let $\mathcal{R}=[-R_1,R_1]\times[-R_2,R_2]\times\ldots\times[-R_n,R_n]$ be a rectangular region centered at the origin in $\mathbb{R}^n$. We assume $\mathcal{R}\subset\Omega$, and let η_i, ϕ_i, ζ_i, $\theta_i\in C_c^\infty\left((-R_i,R_i)\right)$ for $1\leq i\leq n$ satisfy*

1. η_i, ϕ_i and ζ_i equal 1 in a neighborhood of zero,
2. $\zeta_i\succeq\phi_i\succeq\eta_i$ and $\zeta_i\geq\phi_i\geq\eta_i$,
3. $\theta_i\leq 1$, $\theta_i\succeq\eta_i'$, $\theta_i\succeq\phi_i'$ and $\theta_i\succeq\zeta_i'$,
4. 0 does not lie in the support of θ_i.

Set

$$\begin{aligned} \eta\left(x\right)&=\textstyle\prod_{i=1}^n\eta_i\left(x_i\right), \qquad & \phi\left(x\right)&=\textstyle\prod_{i=1}^n\phi_i\left(x_i\right), \\ \zeta\left(x\right)&=\textstyle\prod_{i=1}^n\zeta_i\left(x_i\right), \qquad & \varrho_i\left(x\right)&=\theta_i\left(x_i\right)\textstyle\prod_{j\neq i}\zeta_j\left(x_j\right). \end{aligned}$$

Finally, let $\xi,\varkappa\in C_c^\infty\left(\mathcal{R}\right)$ satisfy $\varkappa\succeq\xi\succeq\zeta$, $\varkappa\geq\xi\geq\zeta$ and $\xi\succeq\varrho_i$.

Let $k^i\left(x\right)$ be smooth and nonnegative in a domain $\Omega\subset\mathbb{R}^n$, $i=1,\ldots,n$. We assume $\vec{k}(x)=\left(k^1(x),\ldots,k^n(x)\right)$ satisfies Hypothesis 1.1, with (1.8) replaced by (1.12), i.e. $k^1(x)\equiv 1$. Moreover, we choose the functions η_i, θ_i above so that the following condition holds:

$$\varrho_i\left(x\right)\,k^i(x,u)\leq C_1\,\varrho_i\left(x\right)\min_{1\leq j\leq n}k^j(x,u), \quad (x,z)\in\Gamma, \quad 1\leq i\leq n, \tag{1.33}$$

for some constant $C_1 = C_1\left(\vec{k}, R_1, \dots, R_n\right) > 0$. This condition can always be met because $\vec{k}$ is Lipschitz and satisfies (1.10).

2. Preliminary results

2.1. **The gradient estimate.**

In this section we establish some estimates on the k-gradient of ∇u for smooth functions u. Our main result is Lemma 2.5, in which $\|\nabla_k \eta \partial u\|_0^2$ is controlled in terms of lower order derivatives and $\mathcal{L}u$, remember that ∂u denotes a generic first order derivative.

We begin by presenting a "Caccioppoli" type inequality for powers of u as in [5], estimating the energy of the $\mathcal{L}$-gradient of powers u^β in terms of $\mathcal{L}u$ and powers of u. We omit the standard proof (see Lemmas 3.1 and 3.9 in [8] for details).

Lemma 2.1. *Suppose $\mathcal{L}$ is as in (1.6), where $\mathbf{A}$ is a nonnegative definite matrix. For $u \in C^\infty(\mathcal{R})$ and $\beta > \frac{1}{2}$, we have*

$$\int_{\mathcal{R}} \left|\zeta \nabla_k u^\beta\right|^2 dx \le \frac{2\beta^2}{2\beta - 1}\left|\int_{\mathcal{R}} (\zeta \mathcal{L}u)\left(\zeta u^{2\beta-1}\right) dx\right| + 2\left(\frac{2\beta}{2\beta-1}\right)^2 \int_{\mathcal{R}} |\nabla_k \zeta|^2 \left|u^\beta\right|^2 dx, \tag{2.1}$$

where $\nabla_k u$ is given by (1.14) in Definition 1.3.

Remark 2.2. *We will often use an absorption technique which is based on the inequality*

$$2\,ab \le \varepsilon\, a^2 + \frac{1}{\varepsilon} b^2, \qquad \varepsilon > 0. \tag{2.2}$$

Since this is a standard technique (used for example in the proof of the lemma above), we will often use it without explicit mention for conciseness.

It will be convenient to set

$$\begin{aligned} A^6 \quad = \quad & 1 + \|\nabla \eta\|_\infty^6 + \|\nabla \phi\|_\infty^6 + \|\nabla \zeta\|_\infty^6 + \|\nabla \varrho_1\|_\infty^6 + \dots + \|\nabla \varrho_n\|_\infty^6 \\ & + \left\|\nabla^2 \eta\right\|_\infty^3 + \left\|\nabla^2 \phi\right\|_\infty^3 + \left\|\nabla^2 \zeta\right\|_\infty^3 \\ & + \left\|\nabla^3 \eta\right\|_\infty^2 + \left\|\nabla^3 \phi\right\|_\infty^2 + \left\|\nabla^3 \zeta\right\|_\infty^2, \end{aligned} \tag{2.3}$$

in order to collect constants in front of the lower order terms in what follows. It is important to observe that since $A \ge R_1^{-1}$, if we wish to show that a certain term is small by applying the one-dimensional Poincaré inequality in the x_1-variable, i.e. by applying the estimate

$$\|\varphi\|_{L^2(\mathbb{R}^n)} \le C R_1 \|\partial_1 \varphi\|_{L^2(\mathbb{R}^n)},$$

where φ is a function with compact support. In order to gain a factor of R_1, we must ensure that the term to be shown small is not multiplied by a constant which increases with A. Note that by Definition 1.3, (1.7) and (1.8), we have, using the one dimensional Poincaré estimate,

$$\|\varphi\|_0 \le C R_1 \|\nabla_k \varphi\|_0. \tag{2.4}$$

Lemma 2.3. *For any smooth function φ and any smooth cutoff function ψ, we have*

$$\int (\mathbf{A}(x, v) \nabla \psi) \nabla \psi \, |\partial \varphi|^2 \le C_1 \Lambda \int |\nabla \psi|^2 |\nabla_{k,w} \varphi|^2,$$

where Λ is as in (1.7), C_1 is given by (1.33) and $\nabla_{k,w}\varphi$ is given by (1.15) in Definition (1.3).

Proof. From (1.7) we have

$$\int \left(\mathbf{A}\left(x,w\right)\nabla\psi\right)\nabla\psi\left|\partial\varphi\right|^{2} \leq \Lambda\sum_{j=1}^{n}\int k_{j}\left(x,w\right)\left|\partial_{j}\psi\right|^{2}\left|\partial\varphi\right|^{2}.$$

In turn, by (1.33) and (1.7) we get

$$\begin{aligned}\int \left(\mathbf{A}\left(x,w\right)\nabla\psi\right)\nabla\psi\left|\partial\varphi\right|^{2} &\leq C_{1}\Lambda\sum_{i,j=1}^{n}\int\left|\partial_{j}\psi\right|^{2}k^{i}\left(x,w\right)\left|\partial_{i}\varphi\right|^{2}\\ &\leq C_{1}\Lambda\sum_{j=1}^{n}\int\left|\partial_{j}\psi\right|^{2}\left|\nabla_{k,w}\varphi\right|^{2}\\ &= C_{1}\Lambda\int\left|\nabla\psi\right|^{2}\left|\nabla_{k,w}\varphi\right|^{2}.\end{aligned}$$

The main lemma of this section establishes control of the L^2-norm of the k-gradient of a derivative of a smooth function u in terms of lower order derivatives and $\mathcal{L}u$.

First, we define parameter the $\kappa\left(u\right)$ as follows

Definition 2.4. *Given a smooth function u, we set $\tilde{\mathbf{A}}_{u}\left(x\right)=\mathbf{A}\left(x,u\left(x\right)\right)$ and define*

$$\kappa\left(u\right)^{2}=\max_{1\leq i\leq n}\sup_{\det\left(\mathbf{A}\left(x,u\left(x\right)\right)\right)\neq0\neq\left|\xi\right|}\frac{\left|\left(\partial_{i}\tilde{\mathbf{A}}_{u}\right)\xi\right|^{2}}{\xi^{t}\mathbf{A}\left(x,u\left(x\right)\right)\xi}.$$

Note that with $\kappa\left(u\right)$ so defined, and φ any smooth cutoff function, we have

$$\left\|\varphi\left(\partial\mathbf{A}\right)\left(\nabla u\right)\right\|_{0}\leq C\kappa\left\|\varphi\nabla_{k}u\right\|_{0}. \tag{2.5}$$

Lemma 2.5. *Suppose $\mathcal{L}$ is as in (1.6) with $\mathbf{A}$ nonnegative, smooth and satisfies (1.7). Let ∂ denote ∂_i for some $1\leq i\leq n$. For $u\in C^{\infty}\left(\mathcal{R}\right)$, we have*

$$\begin{aligned}\left\|\nabla_{k}\eta\partial u\right\|_{0}^{2} \leq\ & CA^{6}\kappa^{2}\left|\int_{\mathcal{R}}\left(\eta\mathcal{L}u\right)\left(\zeta u\right)\right|+C\left|\int_{\mathcal{R}}\left(\eta\partial\mathcal{L}u\right)\left(\eta\partial u\right)\right|\\ &+C\kappa^{2}\left\|\eta\nabla u\right\|_{0}^{2}+CA^{6}\kappa^{2}\left\|\xi u\right\|_{0}^{2},\end{aligned}$$

where $\kappa=\kappa\left(u\right)$ is as in Definition 2.4.

To prove Lemma 2.5, we will first need to establish two auxiliary results.

Lemma 2.6. *For $u\in C^{\infty}\left(\mathcal{R}\right)$ set $\mathcal{L}_{u}=\operatorname{div}\mathbf{A}\left(x,u\right)\nabla$ where $\mathbf{A}$ is nonnegative, smooth and satisfies (1.7) (note that $\mathcal{L}_{u}u=\mathcal{L}u$ where is $\mathcal{L}$ as in (1.6)). Let ∂ denote ∂_i for some $1\leq i\leq n$. Then for $0<\alpha<1$,*

$$\left|\int_{\mathcal{R}}\left(\left[\mathcal{L}_{u},\eta\right]\partial u\right)\left(\eta\partial u\right)\right|\leq\alpha\int_{\mathcal{R}}\left|\nabla_{k,u}\eta\partial u\right|^{2}+\frac{CA^{2}}{\alpha}\int_{\mathcal{R}}\left|\phi\nabla_{k}u\right|^{2},$$

where $\kappa=\kappa\left(u\right)$ is as in Definition 2.4 and $\nabla_{k,u}$ is given by (1.15) in Definition (1.3).

Proof. We compute that

$$\begin{aligned}\left[\mathcal{L}_{u},\eta\right]\partial u &= \operatorname{div}\mathbf{A}\left(x,u\right)\nabla\eta\partial u-\eta\operatorname{div}\mathbf{A}\left(x,u\right)\nabla\partial u\\ &= \operatorname{div}\partial u\mathbf{A}\left(x,u\right)\left(\nabla\eta\right)+\left(\nabla\eta\right)^{t}\mathbf{A}\left(x,u\right)\nabla\partial u\\ &= 2\left(\nabla\eta\right)^{t}\mathbf{A}\left(x,u\right)\nabla\partial u+\left(\partial u\right)\left(\mathcal{L}_{u}\eta\right)\end{aligned}$$

therefore, we have

$$
\begin{aligned}
(2.6)\qquad & \left|\int_{\mathcal{R}}\left(\left[\mathcal{L}_u,\eta\right]\partial u\right)(\eta\partial u)\right| \\
\leq\ & 2\left|\int_{\mathcal{R}}\left((\nabla\partial u)^t\mathbf{A}(x,u)\nabla\eta\right)(\eta\partial u)\right|+\left|\int_{\mathcal{R}}(\partial u)\left(\mathcal{L}_u\eta\right)(\eta\partial u)\right| \\
\leq\ & 3\left|\int_{\mathcal{R}}\left((\eta\nabla\partial u)^t\mathbf{A}(x,u)\nabla\eta\right)(\partial u)\right|+\left|\int_{\mathcal{R}}(\partial u)\left(\nabla\eta\partial u\right)^t\mathbf{A}(x,u)\nabla\eta\right|,
\end{aligned}
$$

where the last inequality was obtained integrating by parts the second term on the second line. Using the identity $\eta\nabla\partial u=\nabla\eta\partial u-(\nabla\eta)\partial u$, it follows that

$$
\begin{aligned}
& \left|\int_{\mathcal{R}}\left(\left[\mathcal{L},\eta\right]\partial u\right)(\eta\partial u)\right| \\
\leq\ & 4\left|\int_{\mathcal{R}}(\partial u)\left(\nabla\eta\partial u\right)^t\mathbf{A}(x,u)\nabla\eta\right|+3\left|\int_{\mathcal{R}}\left((\nabla\eta)^t\mathbf{A}(x,u)\nabla\eta\right)(\partial u)^2\right|, \\
=\ & 4\left|\int_{\mathcal{R}}(\partial u)\left(\nabla\eta\partial u\right)^t\mathbf{A}(x,u)\nabla\eta\right|+\int_{\mathcal{R}}\left|\left(\nabla_{k,u}\eta\right)\partial u\right|^2.
\end{aligned}
$$

By Scwartz inequality we have

$$
\begin{aligned}
2\left|(\nabla\eta\partial u)^t\mathbf{A}(x,u)\nabla\eta\right| &\leq \alpha\left|(\nabla\eta\partial u)^t\mathbf{A}(x,u)(\nabla\eta\partial u)\right|+\frac{1}{\alpha}\left|(\nabla\eta)^t\mathbf{A}(x,u)(\nabla\eta)\right| \\
&= \alpha\left|\nabla_{k,u}\eta\partial u\right|^2+\frac{1}{\alpha}\left|\nabla_{k,u}\eta\right|^2,
\end{aligned}
$$

we apply this inequality and Lemma 2.3 to the first term on the right to obtain

$$
\begin{aligned}
\left|\int_{\mathcal{R}}\left(\left[\mathcal{L},\eta\right]\partial u\right)(\eta\partial u)\right| &\leq \alpha\int_{\mathcal{R}}\left|\nabla_{k,u}\eta\partial u\right|^2+\frac{C}{\alpha}\int_{\mathcal{R}}\left|\left(\nabla_{k,u}\eta\right)\partial u\right|^2 \\
&\leq \alpha\int_{\mathcal{R}}\left|\nabla_{k,u}\eta\partial u\right|^2+\frac{CA^2}{\alpha}\int_{\mathcal{R}}\left|\phi\nabla_k u\right|^2.
\end{aligned}
$$

Lemma 2.7. *Suppose $\mathcal{L}$ is as in (1.6) with k_j nonnegative, smooth and satisfying (1.16), $j=1,\ldots,n$. Let ∂ denote ∂_i for some $1\leq i\leq n$. For $u\in C^\infty(\mathcal{R})$ and $0<\alpha<1$, we have* $\left|\int_{\mathcal{R}}\left(|\nabla\eta|^2\mathcal{L}u\right)(\zeta u)\right|$

$$
\begin{aligned}
\left|\int_{\mathcal{R}}\left(\eta\left[\mathcal{L},\partial\right]u\right)(\eta\partial u)\,dx\right| \leq CA^4\left|\int_{\mathcal{R}}(\eta\mathcal{L}u)(\zeta u)\right|+\frac{C\kappa^2}{\alpha}\left\|\zeta\nabla u\right\|_0^2 \\
+CA^4\left\|\zeta u\right\|_0^2+C\alpha\left\|\nabla_k\eta\partial u\right\|_0^2.
\end{aligned}
$$

where $\kappa=\kappa(u)$ is as in Definition 2.4.

Proof. We have $[\mathcal{L},\partial]=-\operatorname{div}(\partial\mathbf{A})\nabla$ and so, integrating by parts, writing $\left(\nabla\eta^2\partial u\right)=(\eta\nabla\eta\partial u)+((\nabla\eta)\eta\partial u)$, and using (2.5) we obtain

$$
\begin{aligned}
\left|\int_{\mathcal{R}}\left(\eta\left[\mathcal{L},\partial\right]\zeta u\right)(\eta\partial u)\right| &\leq \left|\int_{\mathcal{R}}(\eta\partial u)(\nabla\zeta u)^t(\partial\mathbf{A})(\nabla\eta)\right| \\
&\quad+\left|\int_{\mathcal{R}}\eta(\nabla\zeta u)^t(\partial\mathbf{A})(\nabla\eta\partial u)\right| \\
&\leq CA^2\int_{\mathcal{R}}|\nabla\eta|^2\left|\nabla_k u\right|^2+C\kappa^2\int_{\mathcal{R}}|\eta\nabla u|^2 \\
&\quad+\frac{C\kappa^2}{\alpha}\int_{\mathcal{R}}|\eta\nabla u|^2+C\alpha\int_{\mathcal{R}}\left|\nabla_k\eta\partial u\right|^2
\end{aligned}
$$

We now use (1.17) to write $|\nabla\eta|^2 \leq CA^2\eta$, and Lemma 2.1 to bound $CA^2 \left\| |\nabla\eta| \nabla_k u\right\|_0^2$ by

$$CA^4 \left| \int_{\mathcal{R}} (\eta\mathcal{L}u)(\zeta u)\right| + CA^4 \left\|\zeta u\right\|_0^2,$$

and this completes the proof of the lemma.

Finally, we can give the proof the main result of this section.

Proof of Lemma 2.5. Replacing u^β by $\eta\partial u$ in (2.1) and since $(\nabla_k\zeta)\eta = 0$, we have

$$\left\|\nabla_k\eta\partial u\right\|_0^2 = \int_{\mathcal{R}} |\zeta\nabla_k\eta\partial u|^2 \leq 2\left|\int_{\mathcal{R}} (\mathcal{L}\eta\partial u)(\eta\partial u)\right|.$$

Writing

$$\mathcal{L}\eta\partial u = [\mathcal{L},\eta]\,\partial u + \eta\,[\mathcal{L},\partial]\,u + \eta\partial\mathcal{L}u,$$

and replacing on the above expression, we obtain

$$\begin{aligned}\left\|\nabla_k\eta\partial u\right\|_0^2 &\leq C\left|\int_{\mathcal{R}} ([\mathcal{L},\eta]\,\partial u)(\partial u)\right| + C\left|\int_{\mathcal{R}} (\eta\,[\mathcal{L},\partial]\,u)(\eta\partial u)\right| \\ &\quad + C\left|\int_{\mathcal{R}} (\eta\partial\mathcal{L}u)(\eta\partial u)\right|.\end{aligned}$$

Now we bound the first term on the right by Lemma 2.6, the second term on the right by Lemma 2.7, and collect terms

$$\begin{aligned}\left\|\nabla_k\eta\partial u\right\|_0^2 &\leq C\left(\alpha + R_1^2\right)\int_{\mathcal{R}} |\nabla_k\eta\partial u|^2 + \frac{CA^6\kappa^2}{\alpha}\left|\int_{\mathcal{R}} (\eta\mathcal{L}u)(\zeta u)\right| \\ &\quad + \frac{C\kappa^2}{\alpha}\left\|\eta\nabla u\right\|_0^2 + \frac{CA^6\kappa^2}{\alpha}\left\|\xi u\right\|_0^2 \\ &\quad + C\left|\int_{\mathcal{R}} (\eta\partial\mathcal{L}u)(\eta\partial u)\right|.\end{aligned}$$

Finally, taking α and R_1 small enough, we can absorb the first term on the right into the left to obtain the conclusion of the lemma.

2.2. **The subunit estimate.**

In this section we prove some a priori estimates for the k-gradient of powers of smooth functions u. We show that for all $\beta > 1$, the L^2 norm of $\nabla_k u^\beta$ is controlled in terms of the smoothness of $\mathbf{A}$, $\mathcal{L}u$ and the L^p norm of u^β for some $1 < p < 2$ (note the power gain).

Whenever we use β to denote a positive real number, we assume that $\beta = \frac{m}{n}$ is rational with n odd, so that expressions such as u^β make sense. Let $\mathcal{R} = \prod_{i=1}^n [-R_i, R_i]$ be a rectangle in the plane, and let $\eta, \zeta, \varrho, \xi, \varkappa$ be as in section 1.

We will use the following fractional integral result repeatedly in this effort.

Proposition 2.8. *Suppose T is a pseudodifferential operator of order $\alpha \in (-n, 0]$. Then*

$$\left\|\zeta T\xi f\right\|_{L^q(\mathbb{R}^2)} \leq C\left\|\xi f\right\|_{L^p(\mathbb{R}^n)}, \qquad \frac{1}{q} \geq \frac{1}{p} + \frac{\alpha}{n},$$

provided $1 \leq p \leq q < \infty$, and $q < \frac{n}{n+\alpha}$ in the case $p = 1$. If T is in addition a Fourier multiplier operator, then the cutoff functions ξ and ζ can be omitted.

Lemma 2.9. *Let I_γ denote a Fourier multiplier operator of order $-\gamma$, let ϱ be a smooth nonnegative cutoff function supported in $\mathcal{R}$, and v be a smooth function on $\mathcal{R}$, for any $0<\alpha<1$, and $\varepsilon>0$, there exists $1<p<2$ such that*

$$\begin{aligned}\int_{\mathcal{R}} \left|\nabla_k I_1 \xi v\right|^2 \varrho^2 \leq & \int \left| I_{1-\alpha} \xi \left[\mathcal{L}, I_1\right] \varrho^2 \xi v\right|^2 + \varepsilon \int_{\mathcal{R}} \left|\nabla_k v\right|^2 \varrho^4 \\ & + C\left(A^2 + D^{\frac{5}{2}} + \tfrac{1}{\varepsilon}\right)\left(\int_{\mathcal{R}} \left|\varkappa v\right|^p\right)^{\frac{2}{p}},\end{aligned}$$

where $D \geq 1$ is a bound for $\|\varrho\|_\infty^2 + \|\nabla\varrho\|_\infty + \left\|\nabla^2\varrho\right\|_\infty^{\frac{1}{2}}$, and ϱ furthermore satisfies $dist\,(x,\partial\Omega) \geq D^{-2}$, for all $x \in \operatorname{support}(\varrho)$.

Proof. We set $H = \int_{\mathcal{R}} \left|\nabla_k I_1 \xi v\right|^2 \varrho^2$. Integrating by parts, we have

$$\begin{aligned} H &= -\int_{\mathcal{R}} \left(I_1 \xi v\right)^t \operatorname{div} \mathbf{A}\left(\nabla I_1 \xi v\right) \varrho^2 \\ &\leq \left|\int_{\mathcal{R}} \varrho^2 \left(I_1 \xi v\right)\left(\mathcal{L} I_1 \xi v\right)\right| + \left|\int_{\mathcal{R}} \left(I_1 \xi v\right)\left(\nabla \varrho^2\right)^t \mathbf{A}\left(\nabla I_1 \xi v\right)\right| \\ &= I + II. \end{aligned} \tag{2.7}$$

Note that from (1.17) applied to ϱ^2, we have

$$\left|\partial_i \varrho^2\right| \leq C D^{\frac{5}{4}} \sqrt{\varrho^2},$$

from this, inequality (2.2) and Proposition 2.8, we get

$$\begin{aligned} II &\leq C D^{\frac{5}{2}} \int_{\mathcal{R}} \left|\xi I_1 \xi v\right|^2 + \tfrac{1}{4} H \\ &\leq C D^{\frac{5}{2}} \left(\int_{\mathcal{R}} \left|\xi v\right|^{p_1}\right)^{\frac{2}{p_1}} + \tfrac{1}{4} H, \end{aligned} \tag{2.8}$$

for some $1<p_1<2$. Now, applying the identity

$$I_1 \varrho^2 \mathcal{L} I_1 \xi v = \left[I_1, \varrho^2\right] \mathcal{L} \xi I_1 \xi v + \varrho^2 \left[I_1, \mathcal{L}\right] I_1 \xi v + \varrho^2 \mathcal{L} I_2 \xi v$$

where we used that $\xi \succeq \varrho$, we obtain

$$\begin{aligned} I \leq & \left|\int_{\mathcal{R}} (\xi v)\left(\left[I_1, \varrho^2\right] \operatorname{div} \mathbf{A} \xi \nabla I_1 \xi v\right)\right| + \left|\int_{\mathcal{R}} (\xi v)\left(\varrho^2 \left[I_1, \mathcal{L}\right] I_1 \xi v\right)\right| \\ & + \left|\int_{\mathcal{R}} (\xi v)\left(\varrho^2 \mathcal{L} I_2 \xi v\right)\right| \\ = & \; III + IV + V. \end{aligned} \tag{2.9}$$

Integrating by parts and applying Hölder inequality, we obtain

$$\begin{aligned} III &= \left|\int_{\mathcal{R}} \left(\nabla_k \left[I_1, \varrho^2\right] \xi v\right) \cdot \xi \left(\nabla_k I_1 \xi v\right)\right| \\ &\leq C\left(\int \left|\nabla_k \left[I_1, \varrho^2\right] \xi v\right|^q\right)^{\frac{2}{q}} + C\left(\int_{\mathcal{R}} \left|\xi \nabla_k I_1 \xi v\right|^{q'}\right)^{\frac{2}{q'}}. \end{aligned}$$

And from Proposition 2.8, since $I_1 \xi$ and $\nabla_k \left[I_1, \varrho^2\right]$ have order -1, and $\xi \nabla_k I_1$ have order 0, there exists $1<p_2<2$ and $C=C(n)$ such that for q above satisfying $q>2$ and q close enough to 2, we have

$$III \leq CA^2 \left(\int_{\mathcal{R}} |\xi v|^{p_2} \right)^{\frac{2}{p_2}}. \tag{2.10}$$

For IV, we write

$$IV = \left| \int_{\mathcal{R}} \left(I_{1-\alpha} \left[\mathcal{L}, I_1 \right] \varrho^2 \xi v \right) I_\alpha \left(\xi v \right) \right|,$$

for any $0 < \alpha < 1$, thus,

$$\begin{aligned} IV &\leq \tfrac{1}{2} \int \left| I_{1-\alpha} \xi \left[\mathcal{L}, I_1 \right] \varrho^2 \xi v \right|^2 + \tfrac{1}{2} \int \left| I_\alpha \xi v \right|^2 \\ &\leq \tfrac{1}{2} \int \left| I_{1-\alpha} \xi \left[\mathcal{L}, I_1 \right] \varrho^2 \xi v \right|^2 + C \left(\int_{\mathcal{R}} |\xi v|^{p_3} \right)^{\frac{2}{p_3}}, \end{aligned} \tag{2.11}$$

where $1 < p_3 < 2$, $C = C(n, \alpha)$ after having applied Proposition 2.8 once more. To treat V, we write

$$V = \left| \int_{\mathcal{R}} \left(\nabla_k \varrho^2 \xi v \right)^t \left(\nabla_k I_2 \xi v \right) \right|.$$

Then we use $\left(\nabla_k \varrho^2 \xi v \right) = \left(\nabla_k \varrho^2 \xi \right) v + \left(\nabla_k v \right) \varrho^2 \xi$ and apply Hölder inequality to obtain

$$\begin{aligned} V &\leq \left| \int_{\mathcal{R}} \varrho^2 \xi \left(\nabla_k v \right)^t \left(\nabla_k I_2 \xi v \right) \right| + \left| \int_{\mathcal{R}} \left(\nabla_k \varrho^2 \xi \right)^t v \left(\nabla_k I_2 \xi v \right) \right| \\ &\leq \varepsilon \int_{\mathcal{R}} \varrho^4 \left| \nabla_k v \right|^2 + \frac{C}{\varepsilon} \left(\int_{\mathcal{R}} \varkappa^2 \left| \nabla_k I_2 \xi v \right|^q \right)^{\frac{2}{q}} \\ &\quad + \varepsilon \left(\int_{\mathcal{R}} \left| \left(\nabla_k \varrho^2 \xi \right)^t v \right|^{q'} \right)^{\frac{2}{q'}}, \end{aligned}$$

for any $q > 1$ and $q' = \frac{q}{q-1}$. From Proposition 2.8 on fractional integration, since $\nabla_k I_2$ have order -1, there exists $1 < q' \leq p_4 < 2 < q$ and $C = C(n)$ such that

$$V \leq \varepsilon \int_{\mathcal{R}} \varrho^4 \left| \nabla_k v \right|^2 + \left(\frac{C}{\varepsilon} + \varepsilon \left(A^2 + D^{\frac{5}{2}} \right) \right) \left(\int_{\mathcal{R}} |\varkappa v|^{p_4} \right)^{\frac{2}{p_4}}. \tag{2.12}$$

Collecting the inequalities (2.7), (2.8), (2.9), (2.10) ,(2.11) and (2.12), and since $A,\ D \geq 1 \geq \varepsilon$, we have

$$\begin{aligned} H &\leq \int \left| I_{1-\alpha} \xi \left[\mathcal{L}, I_1 \right] \varrho^2 \xi v \right|^2 + \varepsilon \int_{\mathcal{R}} \varrho^4 \left| \nabla_k v \right|^2 \\ &\quad + C \left(\frac{1}{\varepsilon} + A^2 + D^{\frac{5}{2}} \right) \left(\int_{\mathcal{R}} |\varkappa v|^p \right)^{\frac{2}{p}}, \end{aligned}$$

where $1 < p = \max \left\{ p_1, p_2, p_3, p_4 \right\} < 2$, $C = C(n, \alpha)$ and $0 < \varepsilon \leq 1$.

Lemma 2.10. *Let* I_γ *denote a Fourier multiplier operator of order* $-\gamma$*, then for any* $0 < \nu < 1$ *and*

$$q > \max \left\{ \frac{2}{1 - \frac{\nu}{6}}, n \left(1 + \nu \right) \right\},$$

there exist $0 < \alpha < 1$ *and* $1 < p < 2$ *such that*

$$\int \left| I_{1-\alpha} \xi \left[\mathcal{L}, I_1 \right] \zeta v \right|^2 dx$$

is dominated by

$$C\mathcal{K}^2\left(\int_{\mathcal{R}}|\zeta v|^p\,dx\right)^{\frac{2}{p}},$$

where

$$\mathcal{K}=\|\varkappa\nabla\mathbf{A}\|_{L^q}.$$

Proof. We note that

$$\begin{aligned}[\mathcal{L},I_1] &= \operatorname{div}\mathbf{A}\nabla I_1-\operatorname{div}I_1\mathbf{A}\nabla\\ &= \operatorname{div}\left([\mathbf{A},I_1\nabla]+I_1\nabla^t\mathbf{A}-I_1\mathbf{A}\nabla\right)\\ &= \operatorname{div}\left([\mathbf{A},I_1\nabla]+I_1\left(\nabla^t\mathbf{A}\right)\right),\end{aligned}$$

where if $\mathbf{A}_i$ denotes the i^{th} row of $\mathbf{A}$, $\nabla^t\mathbf{A}$ denotes the column vector given by $(\operatorname{div}\mathbf{A}_1,\dots,\operatorname{div}\mathbf{A}_n)$. Note that

$$([\mathbf{A},I_1\nabla])_i=\sum_{j=1}^n a_{i_j}I_1\partial_j-I_1\partial_j a_{ij}=\sum_{j=1}^n\left[a_{i_j},I_1\partial_j\right].$$

Following [7], we denote by O_I^m the collection of rough pseudodifferential operators mapping $H^{s+m,p}_{\text{compact}}$ to $H^{s,p}_{\text{loc}}$ for $1<p<\infty$ and $s\in I$, where $H^{s,p}$ denotes the Sobolev space of functions whose fractional derivatives up to order s lie in L^p. Now for $0<\mu<1$ and $\varepsilon>0$ we have

$$\varkappa[\mathbf{A},I_1\nabla]\in O^{-\mu}_{(-\varepsilon,\varepsilon)},\qquad\text{for }\mu+\varepsilon<\nu,$$

with norm $\|\varkappa\mathbf{A}\|_{C^\nu(\mathbb{R}^n)}$ by Theorem 4 in [7]. Since $I_{1-\alpha}\xi\operatorname{div}$ has order α, and since $\xi\operatorname{div}=\xi\varkappa\operatorname{div}=\xi\operatorname{div}\varkappa-\xi(\nabla\varkappa)$, we thus have

$$I_{1-\alpha}\xi\operatorname{div}[\mathbf{A},I_1\nabla]\in O^{\alpha-\mu}_{(-\varepsilon,\varepsilon)},\qquad\text{for }\mu+\varepsilon<\nu.$$

Thus $I_{1-\alpha}\xi\operatorname{div}[\mathbf{A},I_1\nabla]$ maps $L^{p_1}_{\text{compact}}=H^{0,p_1}_{\text{compact}}$ to $H^{\mu-\alpha,p_1}$ provided $\mu-\alpha\in(-\varepsilon,\varepsilon)$, i.e. $\mu\in(\alpha-\varepsilon,\alpha+\varepsilon)$. In turn, we have that $H^{\mu-\alpha,p_1}$ is embedded in L^2_{loc} by the Sobolev embedding theorem with $\frac{1}{2}=\frac{1}{p_1}-\frac{\mu-\alpha}{2}$. Note that given $\nu,\mu,\varepsilon>0$, satisfying $\mu+\varepsilon<\nu<1$, we can choose $\alpha>0$ such that $\mu-\varepsilon<\alpha<\mu+\varepsilon$. So all the restrictions on the parameters α, ε and μ are met. Moreover, if we take $\alpha<\mu$, that is, if

$$\mu-\varepsilon<\alpha<\mu<\mu+\varepsilon<\nu<1,\tag{2.13}$$

then

$$\begin{aligned}&\int\left|I_{1-\alpha}\xi\left[\mathcal{L},I_1\right]\zeta v\right|^2\\ \leq\;& C\int\left|I_{1-\alpha}\xi\operatorname{div}[\mathbf{A},I_1\nabla]\zeta v\right|^2+C\int\left|I_{1-\alpha}\xi\operatorname{div}I_1\left(\nabla^t\mathbf{A}\right)\zeta v\right|^2\\ \leq\;& C\|\varkappa k\|^2_{C^\nu}\left(\int_{\mathcal{R}}|\zeta v|^{p_1}\right)^{\frac{2}{p_1}}+C\sum_{i=2}^n\left(\int_{\mathcal{R}}\left|\left(\nabla^t\mathbf{A}\right)\zeta v\right|^{p_2}\right)^{\frac{2}{p_2}},\end{aligned}$$

for $\frac{1}{2}=\frac{1}{p_1}-\frac{\mu-\alpha}{2}$ and $\frac{1}{2}=\frac{1}{p_2}-\frac{1-\alpha}{2}$ by Proposition 2.8. Now, by Hölder inequality, we have that each term on the sum on the right is bounded by

$$\left(\int_{\mathcal{R}}\left|\nabla^t\mathbf{A}\right|^{p_3p_2}\right)^{\frac{2}{p_3p_2}}\left(\int_{\mathcal{R}}|\zeta v|^{p_3'p_2}\right)^{\frac{2}{p_3'p_2}},$$

for $1<p_3<\infty$, and $p_3'=\dfrac{p_3}{p_3-1}$. It remains to show that we can choose p_3, α, ε and μ such that the previous parameter restrictions hold, and moreover

$$p_2p_3 = q, \tag{2.14}$$

$$p_2p_3' < 2. \tag{2.15}$$

From $p_2=\frac{2}{2-\alpha}$, we have that (2.14) holds if, and only if

$$p_3=\frac{q}{p_2}=\frac{q\,(2-\alpha)}{2}>1 \tag{2.16}$$

where the last inequality follows from the hypothesis on q and $\alpha<\nu$. Now,

$$p_2p_3'=\frac{q}{\frac{q}{p_2}-1}=\frac{2}{2-\alpha-\frac{2}{q}}=\frac{2}{2-\alpha-\frac{2}{q}}$$

Then, (2.15) holds if, and only if

$$\alpha<1-\tfrac{2}{q}. \tag{2.17}$$

We choose now

$$\mu=\varepsilon=2\alpha=\frac{\nu}{3}.$$

Then (2.13) is satisfied and (2.17) follows from $q>\frac{2}{1-\nu/6}$. Finally, we note that since $q>n\,(1+\nu)$, by the Sobolev's embedding theorem we have

$$\|\varkappa\mathbf{A}\|_{C^\nu}\le C\,\|\varkappa\nabla\mathbf{A}\|_{L^q}=C\,\mathcal{K}.$$

This completes the proof of the lemma upon taking $p=\max\{p_1,p_3'p_2\}$.

Lemma 2.11. *For each $0<\alpha<1$, there is $1<p<2$ such that for all $u\in C_c^\infty(\mathcal{R})$, $\eta>0$, and $\beta\ge1$,*

$$\int_{\mathcal{R}}\left|\zeta\nabla_k u^\beta\right|^2dx$$

is dominated by

$$C\beta\left|\int_{\mathcal{R}}(\zeta\mathcal{L}u)\left(\zeta u^{2\beta-1}\right)\right|+\mathcal{C}(\alpha)\sum_{i,j=1}^n\int\left|I_{1-\alpha}\xi\,[\mathcal{L},I_{1j}]\,(\partial_i\zeta)^2\,\xi u^\beta\right|^2$$
$$+\mathcal{C}(A,B,\alpha)\left(\int_{\mathcal{R}}\left|\varkappa u^\beta\right|^p\right)^{\frac{2}{p}}.$$

where I_γ, $I_{\gamma,j}$ denote Fourier multipliers operators of order $-\gamma$.

Proof. We use Lemma 2.1, and inequality (2.2) to write

$$\int_{\mathcal{R}}\left|\zeta\nabla_k u^\beta\right|^2\le2\beta\left|\int_{\mathcal{R}}(\zeta\mathcal{L}u)\left(\zeta u^{2\beta-1}\right)\right|+16\int_{\mathcal{R}}|u|^{2\beta}\,|\nabla_k\zeta|^2, \tag{2.18}$$

after we have absorbed the last term into the left-hand side and used $\beta\ge1$. Denote by Λ^s the multiplier operator with symbol $\left(1+|\cdot|^2\right)^{\frac{s}{2}}$. Using the identity

$$Id=\left(I-\nabla^2\right)\Lambda^{-2}=\Lambda^{-2}-\operatorname{div}\nabla\Lambda^{-2}, \tag{2.19}$$

and (1.7), we have

$$\begin{aligned}\int_{\mathcal{R}} |u|^{2\beta} |\nabla_k \zeta|^2 &\leq \int_{\mathcal{R}} \left| \left(\Lambda^{-2} \xi u^\beta \right) \nabla_k \zeta \right|^2 + \int_{\mathcal{R}} \left| \operatorname{div} \nabla \Lambda^{-2} u \right|^{2\beta} |\nabla_k \zeta|^2 \\ &\leq \int_{\mathcal{R}} \left| \left(\Lambda^{-2} \xi u^\beta \right) \nabla_k \zeta \right|^2 + C \sum_{i,j=1}^{n} \int_{\mathcal{R}} k^i \left| \partial_j I_{1,j} \xi u^\beta \right|^2 \cdot |\partial_i \zeta|^2 \\ &= I + II,\end{aligned} \tag{2.20}$$

where $I_{1,j} = \partial_j \Lambda^{-2}$, $j = 1, \ldots, n$. Note that the operators $I_{1,j}$ have order -1 for all $j = 1, \ldots, n$. From the hypothesis on k^i (1.33) and the definition of ζ we have that $k^i |\partial_i \zeta|^2 \leq C\, k_j |\partial_i \zeta|^2$, $1 \leq i, j \leq n$, it follows that the second term on the right of (2.20) is bounded by

$$C \sum_{i,j=1}^{n} \int_{\mathcal{R}} k_j \left| \partial_j I_{1,j} \xi u^\beta \right|^2 |\partial_i \zeta|^2 \leq C \sum_{i,j=1}^{n} \int_{\mathcal{R}} \left| \nabla_k I_{1,j} \xi u^\beta \right|^2 |\partial_i \zeta|^2 .$$

From (1.16) applied to $|\partial_i \zeta|$, we have $|\partial_i \zeta|^2 \leq B^2 \zeta$; and applying Lemma 2.9 with $v = u^\beta$ and $\varrho = \partial_i \zeta$, we obtain

$$\begin{aligned}\int_{\mathcal{R}} |u|^{2\beta} |\nabla_k \zeta|^2 &\leq C \sum_{i,j=1}^{n} \int \left| I_{1-\alpha} \xi \left[\mathcal{L}, I_{1j} \right] (\partial_i \zeta)^2 \xi u^\beta \right|^2 + \varepsilon C B^4 \int_{\mathcal{R}} \left| \zeta \nabla_k u^\beta \right|^2 \\ &\quad + C \left(A^5 + \tfrac{1}{\varepsilon} \right) \left(\int_{\mathcal{R}} \left| \varkappa u^\beta \right|^p \right)^{\frac{2}{p}},\end{aligned}$$

where we used that $D = A^2$, and we applied Proposition 2.8 to the first term on the right of (2.20). Majoring with this inequality the right side of (2.18), and absorbing the second term on the right into the left, we obtain

$$\begin{aligned}\int_{\mathcal{R}} \left| \zeta \nabla_k u^\beta \right|^2 &\leq 4\beta \left| \int_{\mathcal{R}} (\zeta \mathcal{L} u) \left(\zeta u^{2\beta - 1} \right) \right| + C \sum_{i,j=1}^{n} \int \left| I_{1-\alpha} \xi \left[\mathcal{L}, I_{1j} \right] (\partial_i \zeta)^2 \xi u^\beta \right|^2 \\ &\quad + C \left(A^5 + B^4 \right) \left(\int_{\mathcal{R}} \left| \varkappa u^\beta \right|^p \right)^{\frac{2}{p}}.\end{aligned}$$

This concludes the proof of the lemma.

Lemma 2.12. *For each* $\nu > 0$ *and* $q > \max \left\{ \frac{2}{1 - \frac{\nu}{6}}, n(1+\nu) \right\}$, *there exists* $1 < p < 2$ *such that for all* $u \in C_c^\infty(\mathcal{R})$ *and all* $\beta \geq 1$,

$$\int_{\mathcal{R}} |\nabla_k \zeta|^2 \left| u^\beta \right|^2 dx,$$

is dominated by

$$\mathcal{C}(A) \left(\mathcal{K}^2 + \frac{1}{\varepsilon} \right) \left(\int_{\mathcal{R}} \left| \xi u^\beta \right|^p \right)^{\frac{2}{p}} + \varepsilon \int_{\mathcal{R}} \left| \zeta \nabla_k u^\beta \right|^2 ,$$

where $0 < \varepsilon \leq 1$ *is arbitrary and* $\mathcal{K} = \| \varkappa \nabla \mathbf{A} \|_{L^q}$.

Proof. We use the identity (2.19) to write

$$\int_{\mathcal{R}} |\nabla_k \zeta|^2 \left| u^\beta \right|^2 \leq \int_{\mathcal{R}} |\nabla_k \zeta|^2 \left| \Lambda^{-2} \xi u^\beta \right|^2 + \int_{\mathcal{R}} |\nabla_k \zeta|^2 \left| \operatorname{div} \nabla \Lambda^{-2} \xi u^\beta \right|^2 . \tag{2.21}$$

By Lemma 2.3, the second term on the right is bounded by

$$\sum_{i=1}^{n}\int_{\mathcal{R}}\left|\nabla_k\zeta\right|^2\left|\partial_i\left(\partial_i\Lambda^{-2}\xi u^{\beta}\right)\right|^2\leq CA^2\sum_{i,j=1}^{n}\int_{\mathcal{R}}\left|\partial_j\zeta\right|^2\left|\nabla_k\left(I_i\xi u^{\beta}\right)\right|^2,$$

where $I_i=\partial_i\Lambda^{-2}$, $i=1,\ldots,n$, are operators of order -1. Applying Lemma 2.9 with $v=u^{\beta}$ and $\varrho=\partial_j\zeta$, we have that the above sum is dominated by

$$\begin{aligned}CA^2\sum_{i,j=1}^{n}\int\left|I_{1-\alpha}\xi\left[\mathcal{L},I_1\right]\left(\partial_j\zeta\right)^2\xi u^{\beta}\right|^2+CA^2\varepsilon\sum_{j=1}^{n}\int_{\mathcal{R}}\left|\nabla_k u^{\beta}\right|^2\left|\partial_j\zeta\right|^4\\+CA^2\left(A^5+\tfrac{1}{\varepsilon}\right)\left(\int_{\mathcal{R}}\left|\varkappa u^{\beta}\right|^p\right)^{\frac{2}{p}}.\end{aligned}\tag{2.22}$$

where we used that we can take $D=A^2$. In turn, applying Lemma 2.10 to the first term on the right of (2.22) we obtain

$$\begin{aligned}&CA^2\sum_{i=1}^{n}\int_{\mathcal{R}}\left|\nabla\zeta\right|^2\left|\nabla_k\left(I_i\xi u^{\beta}\right)\right|^2\\\leq\ &CA^2\varepsilon\int_{\mathcal{R}}\left|\nabla_k u^{\beta}\right|^2\left|\nabla\zeta\right|^4+CA^2\left(A^5+\mathcal{K}^2+\tfrac{1}{\varepsilon}\right)\left(\int_{\mathcal{R}}\left|\varkappa u^{\beta}\right|^p\right)^{\frac{2}{p}}\end{aligned}\tag{2.23}$$

Now the terms on the first sum on the right of (2.21) satisfies

$$\int_{\mathcal{R}}\left|\nabla_k\zeta\right|^2\left|\Lambda^{-2}\xi u^{\beta}\right|^2\leq CA^2\left(\int_{\mathcal{R}}\left|\xi u^{\beta}\right|^p\right)^{\frac{2}{p}},$$

for any $1<p\leq 2$, by Proposition 2.8 on fractional integration (Λ^{-2} has order α for all $\alpha>-2$). From this inequality, (2.21) and (2.23) we get

$$\int_{\mathcal{R}}\left|\nabla_k\zeta\right|^2\left|u^{\beta}\right|^2\leq CA^2\varepsilon\int_{\mathcal{R}}\left|\nabla_k u^{\beta}\right|^2\left|\nabla\zeta\right|^4+CA^2\left(A^5+\mathcal{K}^2+\tfrac{1}{\varepsilon}\right)\left(\int_{\mathcal{R}}\left|\varkappa u^{\beta}\right|^p\right)^{\frac{2}{p}}$$

Finally, we apply (1.16) to ζ to obtain $\zeta_r^4\leq CA^4\zeta^2$ and replace ε by $A^{-2}\varepsilon$ on the above inequality to finish the proof of the lemma.

3. A Quasilinear Degenerate Subelliptic Equation

In this section we prove Theorem 1.4 for smooth solutions of the quasilinear equation (1.13)

$$\mathcal{L}w=\operatorname{div}\mathbf{A}\left(x,w\right)\nabla w+\vec{g}\left(x,w\right)\cdot\nabla w+f\left(x,w\right)=0,\qquad x\in\Omega.$$

We recall here the *a priori* estimates (1.22) we wish to establish:

$$\left\|\zeta D^{\alpha}w\right\|_{\infty}\leq\mathcal{C}_{\alpha}\left(\left\|\varkappa\nabla w\right\|_{\infty},L\right),\tag{3.1}$$

where w is smooth, satisfies (1.13) and also

$$\left(x,w\left(x\right)\right)\in L\quad\text{for all }x\in\operatorname{support}\left(\varkappa\right).\tag{3.2}$$

We attack the problem by differentiating (1.13), to obtain the equations

$$\begin{aligned}
-\mathcal{L}w_j &= \operatorname{div}\{\partial_j \mathbf{A}(x,w)\}\nabla w + \sum_{i=1}^{n}\left(\partial_j g^i\right) w_i + \sum_{i=1}^{n} g^i\left(\partial_i\partial_j w\right) + \left(\partial_j f\right) \\
&= \operatorname{div}\{\partial_j \mathbf{A}(x,w)\}\nabla w + f_j + f_z w_j \\
&\quad + \sum_{i=1}^{n} g_j^i w_i + \sum_{i=1}^{n} g_z^i w_i w_j + \sum_{i=1}^{n} g^i w_{ij},
\end{aligned} \tag{3.3}$$

for $j = 1, \ldots, n$.

We will apply Lemma 2.5 to the components of ∇w, and we will show that in fact $w \in H^2$, i.e., $\nabla^2 w \in L^2$ *with control*. Since the equations (3.3) are not homogeneous, we must handle with care the terms arising from $\mathcal{L}\nabla w$ in applying Lemma 2.5. We then apply the results of section 2.2 to obtain that $\nabla^2 w \in L^q$ *with control* for q large depending on how small R_1 is chosen, again handling with care the terms arising from $\mathcal{L}\nabla w$.

At this point we repeat the above process with $\nabla^2 w$ in place of ∇w. We apply Lemma 2.5 to the components of $\nabla^2 w$, and using the facts that ∇w is bounded and $\nabla^2 w \in L^q$ *with control*, we show that in fact $w \in H^3$, i.e., $\nabla^3 w \in L^2$ *with control.* From now on, it turns out that due to the nature of the quasilinear systems satisfied by higher order gradients of w, which become progressively less nonlinear, we can continue to alternately apply the reverse Sobolev embedding and the Moser iteration results of section 2.2 to increase the index of smoothness of w that is *under control* by 1 with each repetition. Thus we obtain the *a priori* estimates (3.1).

To handle higher order derivatives, it is convenient to develop a special notation:

Definition 3.1. *Given a smooth functions w and $h = h(x,w)$, and ℓ a nonnegative integer, we denote by $\mathcal{P}_\ell^h\left(\partial w, \partial^2 w, \ldots, \partial^N w\right)$ a generic polynomial with coefficients depending on derivatives of h, and such that each monomial $\prod_{r=1}^{N}\left(\partial^r w\right)^{i_r}$ of $\mathcal{P}_\ell^h$, satisfies*

$$\sum_{r=1}^{N} r\, i_r \leq \ell.$$

With this notation, we have

$$\partial^\ell h(x,w) = \mathcal{P}_\ell^h\left(\partial w, \partial^2 w, \ldots, \partial^\ell w\right).$$

We write $\mathcal{P}_\ell^{h^1,\ldots,h^M}$ when the coefficients of $\mathcal{P}$ depend on more than one function and its derivatives.

3.1. **Reverse Sobolev embedding.**

Here we show that if w is smooth and ∇w satisfies the system (3.3), then $\nabla^2 w \in L^2_{\text{loc}}$.

Theorem 3.2. *Suppose w is a smooth solution of (1.13),*

$$\mathcal{L}w = \operatorname{div}\mathbf{A}(x,w)\nabla w = f(x,w) + \vec{g}(x,w)\cdot\nabla w,$$

in $\mathcal{R}$, f and g^i are smooth and g^i satisfies (1.20) for $i = 1.\ldots.n$. Then $w \in H^2_{\text{loc}}$ and

$$\left\|\zeta\nabla^2 w\right\|_0 \leq \mathcal{C}\left(A, B, C_1, \|\varkappa\nabla w\|_\infty, L\right).$$

We prove this theorem at the end of the section. First, we establish some results that will allow us to handle the nonhomogeneous terms in (3.3). The corollary below is a special case of classic a priori estimates for first derivatives of solutions of divergence form equations, we included this estimates in the appendix (Lemma 5.3).

Lemma 3.3. *Suppose w is a smooth solution of equation (1.13) in a compact rectangle $\mathcal{R}$ in Ω'. Then we have*

$$\int_{\mathcal{R}} \left|\nabla_k \zeta w_j\right|^2 dx \leq CB^2 \int_{\mathcal{R}} \left\{\left|\xi \nabla w\right|^3 + 1\right\} + CA^4 C_1^2. \tag{3.4}$$

$1 \leq j \leq n$, where B is as in (1.16) and C_1 is given in (1.33).

Proof. From Lemma 2.1, applied with $k\left(x, w\left(x\right)\right)$ in place of $k\left(x\right)$ there, we have

$$\int_{\mathcal{R}} \left|\nabla_k \zeta w_j\right|^2 dx \leq C \left|\int_{\mathcal{R}} \left(\zeta \mathcal{L} w_j\right)\left(\zeta w_j\right)\right| + C \int_{\mathcal{R}} \left|\nabla_k \zeta\right|^2 \left|w_j\right|^2. \tag{3.5}$$

For the integral involving $\mathcal{L} w_j$, we have, by (3.3) and integration by parts that

$$\begin{aligned}\int_{\mathcal{R}} \left(\zeta \mathcal{L} w_j\right)\left(\zeta w_j\right) &= \int_{\mathcal{R}} \left(\nabla w\right)^t \left(\left\{\partial_j \mathbf{A}\left(x, w\right)\right\} \nabla \zeta^2 w_j\right) + \int_{\mathcal{R}} \left(\zeta f_j\right)\left(\zeta w_j\right) \\ &\quad + \int_{\mathcal{R}} \left(\zeta f_z w_j\right)\left(\zeta w_j\right) + \sum_{i=1}^n \int_{\mathcal{R}} \left(\zeta g_j^i w_i\right)\left(\zeta w_j\right) \\ &\quad + \sum_{i=1}^n \int_{\mathcal{R}} \left(\zeta g_z^i w_i w_j\right)\left(\zeta w_j\right) + \sum_{i=1}^n \int_{\mathcal{R}} \left(\zeta g^i \partial_i w_j\right)\left(\zeta w_j\right).\end{aligned}$$

Using (1.19), (1.20) and inequality (2.2) then yields

$$\left|\int_{\mathcal{R}} \left(\zeta \mathcal{L} w_j\right)\left(\zeta w_j\right)\right| = \frac{CB^2}{\alpha} \int_{\mathcal{R}} \left\{\left|\xi \nabla w\right|^3 + 1\right\} + CA^2 \int_{\mathcal{R}} \left|\xi \nabla_k w\right|^2 + C\alpha \int_{\mathcal{R}} \left|\nabla_k \zeta w_j\right|^2. \tag{3.6}$$

On the other hand, from (1.7) and (1.33),

$$\begin{aligned}\int_{\mathcal{R}} \left|\nabla_k \zeta\right|^2 \left|w_j\right|^2 &\leq C \int_{\mathcal{R}} \sum_{i=1}^n k^i \left|\partial_i \zeta\right|^2 \left|w_j\right|^2 \\ &\leq CC_1^2 \int_{\mathcal{R}} \sum_{i,j=1}^n \left|\partial_i \zeta\right|^2 k_j \left|w_j\right|^2 \\ &\leq CA^2 C_1^2 \int_{\mathcal{R}} \left|\xi \nabla_k w\right|^2.\end{aligned}$$

Now we replace this inequality and (3.6) into the right side of (3.5) to obtain

$$\int_{\mathcal{R}} \left|\nabla_k \zeta w_j\right|^2 dx \leq \frac{CB^2}{\alpha} \int_{\mathcal{R}} \left\{\left|\xi \nabla w\right|^3 + 1\right\} + CA^2 C_1^2 \int_{\mathcal{R}} \left|\xi \nabla_k w\right|^2 + C\alpha \int_{\mathcal{R}} \left|\nabla_k \zeta w_j\right|^2. \tag{3.7}$$

On the other hand, from Lemma 2.1 and (1.20)

$$\begin{aligned}\int_{\mathcal{R}} \left|\zeta \nabla_k w\right|^2 &\leq C \left|\int_{\mathcal{R}} \left(\zeta f\left(x, w\right) + \zeta \vec{g}\left(x, z\right) \cdot \nabla w\right)\left(\zeta w\right)\right| + C \int_{\mathcal{R}} \left|\nabla_k \zeta\right|^2 \left|w\right|^2 \\ &\leq C\alpha \int_{\mathcal{R}} \left|\zeta \nabla_k w\right|^2 + \frac{CA^2}{\alpha}.\end{aligned}$$

Absorbing the last term into the left shows that $\int_{\mathcal{R}} \left|\zeta \nabla_k w\right|^2$ is bounded by CA^2. From this and (3.7)

$$\int_{\mathcal{R}} \left|\nabla_k \zeta w_j\right|^2 dx \leq CB^2 \int_{\mathcal{R}} \left\{\left|\xi \nabla w\right|^3 + 1\right\} + CA^4 C_1^2.$$

The lemma above shows that when w is a solution of (1.13), the k-gradient of ∇w is controlled locally by the L^4 norm of ∇w and $\left\|w\right\|_{\infty}$. In the next lemma we establish some similar a priori control for the k-gradient of $\nabla^2 w$.

Lemma 3.4. *Suppose w is a smooth solution of (1.13) in $\mathcal{R}$. Then, for any $\gamma > 0$ we have*

$$\begin{aligned}\left|\int_{\mathcal{R}} \left(\mathcal{L}\eta w_j\right)\left(\eta w_j\right)\right| \leq \frac{CA^4}{\gamma}\int_{\mathcal{R}} \left|\zeta\nabla w\right|^2 + CB^2\int_{\mathcal{R}}\left|\zeta\nabla w\right|^3 \\ +\gamma\int_{\mathcal{R}}\left|\eta\nabla^2 w\right|^2 + CA^4B^2C_1^2.\end{aligned}$$

Proof. From $\mathcal{L}\eta = \eta\mathcal{L} + [\mathcal{L},\eta]$ we write

$$\left|\int_{\mathcal{R}} \left(\mathcal{L}\eta w_j\right)\left(\eta w_j\right)\right| \leq \left|\int_{\mathcal{R}} \left(\eta\mathcal{L} w_j\right)\left(\eta w_j\right)\right| + \left|\int_{\mathcal{R}} \left(\left[\mathcal{L},\eta\right] w_j\right)\left(\eta w_j\right)\right|. \tag{3.8}$$

By (3.3) and integration by parts, we get

$$\begin{aligned}\left|\int_{\mathcal{R}} \left(\eta\mathcal{L} w_j\right)\left(\eta w_j\right)\right| &\leq \left|\int_{\mathcal{R}} \left(\left(\partial_j\mathbf{A}\right)\nabla w + w_j\left(\partial_z\mathbf{A}\right)\right)^t\left(\nabla\eta w_j\right)\right| \\ &\quad + \left|\int_{\mathcal{R}} \eta\left(f_j + \sum_{i=1}^n g_j^i w_i\right)\left(\eta w_j\right)\right| + \\ &\quad + \left|\int_{\mathcal{R}} \eta\left(f_z + \sum_{i=1}^n g_z^i w_i\right)\left(\eta w_j^2\right)\right| \\ &\quad + \left|\int_{\mathcal{R}} \left(\eta\sum_{i=1}^n g^i w_{ij}\right)\left(\eta w_j\right)\right|.\end{aligned}$$

Since f, $\vec{g}$ and $\mathbf{A}$ are Lipschitz, by Schwartz inequality we have

$$\begin{aligned}\left|\int_{\mathcal{R}} \left(\eta\mathcal{L} w_j\right)\left(\eta w_j\right)\right| &\leq \frac{CA^2}{\gamma}\int_{\mathcal{R}}\left|\zeta\nabla w\right|^2 + C\int_{\mathcal{R}}\left|\zeta\nabla w\right|^3 \\ &\quad +\gamma\int_{\mathcal{R}}\left|\eta\nabla^2 w\right|^2 + C.\end{aligned} \tag{3.9}$$

On the other hand, since

$$\begin{aligned}\left[\mathcal{L},\eta\right] &= \operatorname{div}\mathbf{A}\nabla\eta - \eta\operatorname{div}\mathbf{A}\nabla \\ &= \nabla^t\mathbf{A}\left(\nabla\eta\right) + \left(\nabla\eta\right)^t\mathbf{A}\nabla,\end{aligned} \tag{3.10}$$

where we wrote $\operatorname{div} = \nabla^t$. We have

$$\begin{aligned}&\left|\int_{\mathcal{R}} \left(\left[\mathcal{L},\eta\right] w_j\right)\left(\eta w_j\right)\right| \\ &\leq \left|\int_{\mathcal{R}} \left(\left(\nabla\eta\right) w_j\right)\left(A\nabla\eta w_j\right)\right| + \left|\int_{\mathcal{R}} \left(\left(\nabla\eta\right)^t A\nabla w_j\right)\left(\eta w_j\right)\right| \\ &\leq CA^4\int_{\mathcal{R}}\left|\zeta w_j\right|^2 + C\int_{\mathcal{R}}\left|\nabla_k\eta w_j\right|^2 \\ &\leq CA^4\int_{\mathcal{R}}\left|\zeta w_j\right|^2 + CB^2\int_{\mathcal{R}}\left\{\left|\zeta\nabla w\right|^3 + 1\right\} + CA^4C_1^2,\end{aligned} \tag{3.11}$$

where the last inequality follows from Lemma 3.3. Then the lemma follows from (3.8), (3.9) and (3.11).

Lemma 3.5. *Suppose w is a smooth solution of (1.13) in $\mathcal{R}$. Then we have*

$$\begin{aligned}\left|\int_{\mathcal{R}}(\eta\partial\mathcal{L}\eta w_j)(\partial\eta w_j)\right| &\le \frac{CB^2}{\alpha}\int_{\mathcal{R}}\left\{1+|\zeta\nabla w|^4\right\}\left|\eta\nabla^2 w\right|^2\\ &+\frac{\mathcal{C}(A,B,C_1)}{\alpha}\int_{\mathcal{R}}\left\{1+|\zeta\nabla w|^8\right\}+C\alpha\int_{\mathcal{R}}|\nabla_k\partial\eta w_j|^2\end{aligned}$$

Proof. First, we commute η and $\mathcal{L}$ to obtain

$$\left|\int_{\mathcal{R}}(\eta\partial\mathcal{L}\eta w_j)(\partial\eta w_j)\right| \le \left|\int_{\mathcal{R}}(\eta\partial\eta\mathcal{L}w_j)(\partial\eta w_j)\right|+\left|\int_{\mathcal{R}}(\eta\partial\left[\mathcal{L},\eta\right]w_j)(\partial\eta w_j)\right|. \tag{3.12}$$

From (3.3) and integration by parts, we get

$$\begin{aligned}&\left|\int_{\mathcal{R}}(\eta\partial\eta\mathcal{L}w_j)(\partial\eta w_j)\right|\\ \le\ &\left|\int_{\mathcal{R}}(\eta\partial\eta\operatorname{div}\{\partial_j\mathbf{A}(x,w)\}\nabla w)(\partial\eta w_j)\right|\\ &+\left|\int_{\mathcal{R}}\left(\eta\partial\eta\left\{f_j+f_zw_j+\sum_{i=1}^n g_j^i w_i\right\}\right)(\partial\eta w_j)\right|\\ &+\left|\sum_{i=1}^n\int_{\mathcal{R}}\left(\eta\partial\eta g_z^i w_i w_j\right)(\partial\eta w_j)\right|+\left|\sum_{i=1}^n\int_{\mathcal{R}}\left(\eta\partial\eta g^i\, w_{ij}\right)(\partial\eta w_j)\right|\\ =\ &I+II+III+IV.\end{aligned}\tag{3.13}$$

To treat I, since $\{\partial_j\mathbf{A}(x,w)\}=(\partial_j\mathbf{A})+w_j(\partial_z\mathbf{A})$, we only need to consider

$$\tilde{I}=\left|\int_{\mathcal{R}}(\eta\partial\eta\operatorname{div}w_j(\partial_z\mathbf{A})\nabla w)(\partial\eta w_j)\right|,$$

since this term has the *highest order*. Commuting η with ∂, and then commuting $(\partial\eta)\eta$ and η^2 with the divergence operator, we write

$$\begin{aligned}&\eta\partial\eta\operatorname{div}w_j(\partial_z\mathbf{A})\nabla w\\ =\ &-(\partial\eta)\eta\operatorname{div}w_j(\partial_z\mathbf{A})\nabla w+\partial\eta^2\operatorname{div}w_j(\partial_z\mathbf{A})\nabla w\\ =\ &-(\nabla(\partial\eta)\eta)^t w_j(\partial_z\mathbf{A})\nabla w-\operatorname{div}(\partial\eta)\eta w_j(\partial_z\mathbf{A})\nabla w\\ &+\partial\left(\nabla\eta^2\right)^t w_j(\partial_z\mathbf{A})\nabla w+\operatorname{div}\partial\eta^2 w_j(\partial_z\mathbf{A})\nabla w.\end{aligned}$$

From this, the triangle inequality, and integration by parts:

$$\begin{aligned}\tilde{I}\ \le\ &\left|\int_{\mathcal{R}}\left((\nabla(\partial\eta)\eta)^t w_j(\partial_z\mathbf{A})\nabla w\right)(\partial\eta w_j)\right|\\ &+\left|\int_{\mathcal{R}}((\partial\eta)\eta w_j\nabla w)^t((\partial_z\mathbf{A})\nabla\partial\eta w_j)\right|\\ &+\left|\int_{\mathcal{R}}\left(\left(\nabla\eta^2\right)^t w_j(\partial_z\mathbf{A})\nabla w\right)\left(\partial^2\eta w_j\right)\right|\\ &+\left|\int_{\mathcal{R}}\left(\partial\eta^2 w_j(\partial_z\mathbf{A})\nabla w\right)^t(\nabla\partial\eta w_j)\right|\\ =\ &V+VI+VII+VIII.\end{aligned}\tag{3.14}$$

Then

$$V\le CA^4\int_{\mathcal{R}}|\zeta\nabla w|^4+C\int_{\mathcal{R}}|\partial\eta w_j|^2. \tag{3.15}$$

And by (1.20) we have

$$VI \leq \frac{CB^2A^2}{\alpha}\int_{\mathcal{R}}|\zeta\nabla w|^4 + C\alpha\int_{\mathcal{R}}|\nabla_k\partial\eta w_j|^2 . \tag{3.16}$$

Also, since by (1.20) and (1.33) we have

$$\left|\partial_z\mathbf{A}\left(\nabla\eta^2\right)\right| \leq B\left(\sum_{i=1}^{n}k^i\left(\partial_i\eta^2\right)^2\right)^{\frac{1}{2}} \leq BC_1\sqrt{k_\nu}\left|\nabla\eta^2\right| ,$$

for any $1\leq\nu\leq n$, then writing $\partial=\partial_\nu$:

$$\begin{aligned} VII &\leq \int_{\mathcal{R}}|\zeta\nabla w|^2\left|\partial_z\mathbf{A}\left(\nabla\eta^2\right)\right|\left|\partial^2\eta w_j\right| \\ &\leq BC_1\int_{\mathcal{R}}|\zeta\nabla w|^2\left|\nabla\eta^2\right|\left|\sqrt{k_\nu}\partial_\nu\partial\eta w_j\right| \\ &\leq \frac{CA^2B^2C_1^2}{\alpha}\int_{\mathcal{R}}|\zeta\nabla w|^4 + C\alpha\int_{\mathcal{R}}|\nabla_k\partial\eta w_j|^2 . \end{aligned} \tag{3.17}$$

Operating ∂ on the left factor of the integrand in IX, and applying (1.20) we get

$$\begin{aligned} VIII &\leq \left|\int_{\mathcal{R}}\left(\left(\partial\eta^2\right)w_j\nabla w\right)^t\left(\left(\partial_z\mathbf{A}\right)\nabla\partial\eta w_j\right)\right| \\ &\quad+\left|\int_{\mathcal{R}}\left(\eta^2\left(\partial w_j\nabla w\right)^t\right)\left(\left(\partial_z\mathbf{A}\right)\nabla\partial\eta w_j\right)\right| \\ &\quad+\left|\int_{\mathcal{R}}\left(\operatorname{div}\eta^2w_j\left(\partial\partial_z\mathbf{A}\right)\nabla w\right)\left(\partial\eta w_j\right)\right| \\ &\leq \frac{CB^2A^2}{\alpha}\int_{\mathcal{R}}|\zeta\nabla w|^4 + \frac{CB^2}{\alpha}\int_{\mathcal{R}}|\eta\nabla w|^2\left|\eta\nabla^2 w\right|^2 \\ &\quad+C\alpha\int_{\mathcal{R}}|\nabla_k\partial\eta w_j|^2 + C\int_{\mathcal{R}}|\partial\eta w_j|^2 \\ &\quad+C\int_{\mathcal{R}}\left|\operatorname{div}\eta^2w_j\left(\partial\partial_z\mathbf{A}\right)\nabla w\right|^2 . \end{aligned}$$

Since $\left(\partial\left(\partial_z\mathbf{A}\right)\right)=\left(\partial_\nu\partial_z\mathbf{A}\right)+w_\nu\left(\partial_z^2\mathbf{A}\right)$, where we wrote $\partial=\partial_\nu$, for some index $1\leq\nu\leq n$, calculating the divergence on the last term of the right, we see that

$$\begin{aligned} \int_{\mathcal{R}}\left|\operatorname{div}\eta^2w_j\left(\partial\partial_z\mathbf{A}\right)\nabla w\right|^2 &\leq CA^2\int_{\mathcal{R}}\left\{|\zeta\nabla w|^4+|\zeta\nabla w|^6\right\}+C\int_{\mathcal{R}}|\zeta\nabla w|^8 \\ &\quad+C\int_{\mathcal{R}}\left|\eta\nabla^2 w\right|^2\left\{|\zeta\nabla w|^2+|\zeta\nabla w|^4\right\} . \end{aligned}$$

Replacing back into VII gives

$$\begin{aligned} VII &\leq \frac{CB^2A^2}{\alpha}\int_{\mathcal{R}}\left\{1+|\zeta\nabla w|^8\right\} \\ &\quad+\frac{CB^2}{\alpha}\int_{\mathcal{R}}\left\{1+|\zeta\nabla w|^4\right\}\left|\eta\nabla^2 w\right|^2 \\ &\quad+C\alpha\int_{\mathcal{R}}|\nabla_k\partial\eta w_j|^2 . \end{aligned} \tag{3.18}$$

Applying the estimates (3.15), (3.16), (3.17) and (3.18) to (3.14) yields

$$
\begin{aligned}
\tilde{I} \;\leq\; & \frac{CB^2}{\alpha}\int_{\mathcal{R}}\left\{1+|\zeta\nabla w|^4\right\}\left|\eta\nabla^2 w\right|^2 \\
&+\frac{CA^4B^2C_1^2}{\alpha}\int_{\mathcal{R}}\left\{1+|\zeta\nabla w|^8\right\} \\
&+C\alpha\int_{\mathcal{R}}\left|\nabla_k\partial\eta w_j\right|^2 .
\end{aligned} \tag{3.19}
$$

The terms II and III in (3.13) are handled in a similar fashion. Indeed, developing the derivatives in the first factor of the integrands, we have

$$
\begin{aligned}
II+III \;=\; & \left|\int_{\mathcal{R}}\left(\eta\left\{\partial\eta f_j+\partial f_z\eta w_j+\sum_{i=1}^{n}\partial g_j^i\left(x,z\right)\eta w_i\right\}\right)\left(\partial\eta w_j\right)\right| \\
&+\left|\int_{\mathcal{R}}\left(\eta\partial\eta\sum_{i=1}^{n}g_z^i\left(x,z\right)w_iw_j\right)\left(\partial\eta w_j\right)\right| \\
\leq\; & C\int_{\mathcal{R}}\left\{\eta\left|\mathcal{P}_3^{\vec{g},f}\left(\nabla w\right)\right|+\zeta\left|\mathcal{P}_1^{\vec{g},f}\left(\nabla w\right)\right|\sum_{i=1}^{n}\left|\partial\eta w_i\right|\right\}\left|\partial\eta w_j\right| \\
&+CA\int_{\mathcal{R}}\left\{\zeta\left|\mathcal{P}_2^{\vec{g},f}\left(\nabla w\right)\right|+1\right\}\left|\partial\eta w_j\right| \\
\leq\; & CA^2\int_{\mathcal{R}}\left\{|\zeta\nabla w|^6+1\right\}+C\int_{\mathcal{R}}\left\{|\zeta\nabla w|^2+1\right\}\left|\eta\nabla^2 w\right|^2 .
\end{aligned} \tag{3.20}
$$

where $\mathcal{P}_m^{\vec{g},f}$ is a polynomial of degree m with smooth coefficients depending on the derivatives of f and $\vec{g}$, as given in Definition 3.1. For IV, we write

$$
\partial\eta g^i\,w_{ij}=\left(\partial\eta\right)g^i\,w_{ij}+\eta\left(\partial g^i\right)\partial_i w_j+\eta g^i\,\partial_i\zeta\partial w_j,
$$

where we used that $\eta\partial_i=\eta\partial_i\zeta$. We replace this equality into V and use the triangle inequality to obtain

$$
\begin{aligned}
IV \;\leq\; & \left|\int_{\mathcal{R}}\left(\sum_{i=1}^{n}\eta\left(\partial\eta\right)g^i\partial_i w_j\right)\left(\zeta\partial\eta w_j\right)\right|+\left|\int_{\mathcal{R}}\left(\sum_{i=1}^{n}\eta^2\left(\partial g^i\right)\partial_i w_j\right)\left(\zeta\partial\eta w_j\right)\right| \\
&+\left|\int_{\mathcal{R}}\left(\sum_{i=1}^{n}\eta^2 g^i\partial_i\zeta\partial w_j\right)\left(\zeta\partial\eta w_j\right)\right| .
\end{aligned}
$$

From (1.20) and $\left|\partial g^i\right|=\left|g_\nu^i+g_z^i w_\nu\right|\leq C\left(1+|\nabla w|\right)$, we obtain

$$
\begin{aligned}
IV \;\leq\; & CA^2\int_{\mathcal{R}}\left|\zeta\nabla_k w_j\right|^2+\frac{C}{\alpha}\int_{\mathcal{R}}\left|\partial\eta w_j\right|^2 \\
&+C\int_{\mathcal{R}}\left(1+|\zeta\nabla w|\right)^2\left|\eta\nabla^2 w\right|^2+C\alpha\int_{\mathcal{R}}\left|\nabla_k\partial w_j\right|^2 .
\end{aligned} \tag{3.21}
$$

Collecting the estimates (3.19), (3.20) and (3.21), and replacing into (3.13) yields

$$\begin{aligned}&\left|\int_{\mathcal{R}}(\eta\partial\eta\mathcal{L}w_j)(\zeta\partial\eta w_j)\right|\\ \leq\ &\frac{CB^2}{\alpha}\int_{\mathcal{R}}\left\{1+|\zeta\nabla w|^4\right\}\left|\eta\nabla^2 w\right|^2+\frac{CA^4B^2C_1^2}{\alpha}\int_{\mathcal{R}}\left\{1+|\zeta\nabla w|^8\right\}\\ &+CC_1^2B^2A^2\int_{\mathcal{R}}|\eta\nabla_k w_j|^2+C\alpha\int_{\mathcal{R}}|\nabla_k\partial\eta w_j|^2.\end{aligned}$$

Applying Lemma 3.3 to the third term on the right yields

$$\begin{aligned}&\left|\int_{\mathcal{R}}(\eta\partial\eta\mathcal{L}w_j)(\zeta\partial\eta w_j)\right|\\ \leq\ &\frac{CB^2}{\alpha}\int_{\mathcal{R}}\left\{1+|\zeta\nabla w|^4\right\}\left|\eta\nabla^2 w\right|^2+\frac{CA^4B^4C_1^2}{\alpha}\int_{\mathcal{R}}\left\{1+|\zeta\nabla w|^8\right\}\\ &+CA^6B^2C_1^4+C\alpha\int_{\mathcal{R}}|\nabla_k\partial\eta w_j|^2.\end{aligned}\tag{3.22}$$

Now, we apply (3.10) to the second term in (3.12) to obtain

$$\begin{aligned}&\left|\int_{\mathcal{R}}(\eta\partial[\mathcal{L},\eta]w_j)(\partial\eta w_j)\right|\\ \leq\ &\left|\int_{\mathcal{R}}(\partial\mathbf{A}(\nabla\eta)w_j)(\nabla\eta\partial\eta w_j)\right|+\left|\int_{\mathcal{R}}\left(\eta\partial(\nabla\eta)^t\mathbf{A}\nabla w_j\right)(\partial\eta w_j)\right|.\end{aligned}\tag{3.23}$$

The, applying the previous techniques we see that (3.23) is bounded by the right-hand side of (3.22). Then the lemma follows from this observation, (3.22) and (3.12).

Lemma 3.6. *Suppose w is a smooth solution of (1.13) in $\mathcal{R}$, where $k(x,z)$ is nonnegative and smooth in Ω and satisfies (1.16) so that $w_j=w_{x_j}$, $1\leq j\leq n$, are smooth solutions in $\mathcal{R}$ of the nonlinear system (3.3). Then we have*

$$\begin{aligned}\|\nabla_k\partial\eta w_j\|_0^2\ \leq\ &CB^2\int_{\mathcal{R}}\left\{1+|\zeta\nabla w|^4\right\}\left|\eta\nabla^2 w\right|^2\\ &+\mathcal{C}(A,B,C_1)\left(1+\|\varkappa\nabla w\|_\infty\right)^4\int_{\mathcal{R}}\left\{1+|\zeta\nabla w|^8\right\}.\end{aligned}\tag{3.24}$$

Proof. We apply Lemma 2.5 with $k(x)$ replaced by $k=k(x,w(x))$. First, by (1.18), we have

$$\begin{aligned}|(\partial_i\mathbf{A}(x,w(x)))\xi|^2\ &=\ |(\partial_i\mathbf{A})\xi+w_i(\partial_z\mathbf{A})\xi|^2\\ &\leq\ CB^2(1+\|\nabla w\|_\infty)^2\left(\xi^t\mathbf{A}(x,w)\xi\right)\end{aligned}\tag{3.25}$$

and thus we can bound $\kappa=\kappa(w)$ in Lemma 2.5 by

$$\kappa\leq\widetilde{B}=CB\left(1+\|\varkappa\nabla w\|_\infty\right),\tag{3.26}$$

we obtain

$$\begin{aligned}\|\nabla_k\partial\eta w_j\|_0^2\ \leq\ &CA^4\kappa^2\left|\int_{\mathcal{R}}(\mathcal{L}\eta w_j)(\eta w_j)\right|+C\left|\int_{\mathcal{R}}(\partial\mathcal{L}\eta w_j)(\eta\partial\eta w_j)\right|\\ &+C\kappa^2\|\eta\nabla u\|_0^2+CA^6\kappa^2\|\xi u\|_0^2\end{aligned}$$

In turn, applying Lemmas (3.4) and (3.5) to the first two terms on the right, respectively, yields

$$\begin{aligned}\left\|\nabla_k\partial\eta w_j\right\|_0^2 \leq\ & \frac{CB^2}{\alpha}\int_{\mathcal{R}}\left\{1+\left|\zeta\nabla w\right|^4\right\}\left|\eta\nabla^2 w\right|^2+\kappa^4\frac{\mathcal{C}\left(A,B,C_1\right)}{\alpha}\int_{\mathcal{R}}\left\{1+\left|\zeta\nabla w\right|^8\right\}\\ &+C\alpha\int_{\mathcal{R}}\left|\nabla_k\partial\eta w_j\right|^2,\end{aligned}$$

where we took $\gamma=A^{-4}\kappa^{-2}$. The lemma follows from (3.26) and absorbing the last term on the right into the left.

Proof of Theorem 3.2. The Poincaré inequality and Lemma 3.6 yield with $w_j=\partial_j w$, $j=1,\ldots,n$,

$$\begin{aligned}\int_{\mathcal{R}}\left|\eta\nabla^2 w\right|^2 dx &\leq \sum_{i,j=1}^{n}\int_{\mathcal{R}}\left|\partial_i\eta w_j\right|^2+CA^2\left\|\varkappa\nabla w\right\|_\infty^2\\ &\leq R_1^2\sum_{i,j=1}^{n}\int_{\mathcal{R}}\left|\nabla_k\left(\partial_i\eta w_j\right)\right|^2+CA^2\left\|\varkappa\nabla w\right\|_\infty^2\\ &\leq R_1^2\mathcal{C}\left(B,\left\|\varkappa\nabla w\right\|_\infty\right)\int_{\mathcal{R}}\left|\eta\nabla^2 w\right|^2+R_1^2\mathcal{C}\left(A,B,C_1,\left\|\varkappa\nabla w\right\|_\infty\right).\end{aligned}$$

Choosing $R_1\leq\left\{2\mathcal{C}\left(B,\left\|\varkappa\nabla w\right\|_\infty\right)\right\}^{-1}$ (note that A is not involved here) permits the first term on the right above to be absorbed into the left-hand side, and this completes the proof of the theorem.

3.2. An L^p improvement.

In this subsection, we improve the index of smoothness of w that is a priori under control by showing that $\nabla^2 w\in L^q_{\text{loc}}$ for large $q>2$, with a priori control.

Definition 3.7. *We first define a set of notational* abuses *that make calculations more clear, and we will use henceforth. We denote by ∂ a derivative with respect to x_i for some $1\leq i\leq n$, not necessarily the same for each occurrence of ∂. In this way $D^\alpha w=\partial^{|\alpha|}w$ for any multi-index $\alpha=(\alpha_1.\ldots,\alpha_n)$.*

Theorem 3.8. *Suppose that w solves (1.13) so that $w_i=w_{x_i}$ $i=1,\ldots,n$, give a smooth solution of the system (3.3) in $\mathcal{R}$. Then for $q>1$, we have $w_{ij},\in L^q_{\text{loc}}$, $1\leq i,j\leq n$, and*

$$\left\|\zeta\nabla^2 w\right\|_{L^q}\leq\mathcal{C}\left(\left\|\varkappa\nabla w\right\|_\infty,L\right),$$

provided R_1 is sufficiently small, depending on q.

Next, we have a technical lemma for second derivatives of solutions.

Lemma 3.9. *Suppose that w is a smooth solution of (1.13) in $\mathcal{R}$. Then for $\beta\geq 1$, the integral*

$$\left|\int_{\mathcal{R}}\left(\zeta\mathcal{L}\partial^2 w\right)\left(\zeta\left(\partial^2 w\right)^{2\beta-1}\right)\right|$$

is dominated by

$$\begin{aligned}&C\left|\int_{\mathcal{R}}\left(\left(\partial^2\mathbf{A}\right)\nabla w\right)\cdot\nabla\zeta^2\left(\partial^2 w\right)^{2\beta-1}\right|\\&+C\int_{\mathcal{R}}\left|\left(\partial\mathbf{A}\right)\nabla\zeta\right|^2\left|\partial^2 w\right|^{2\beta}+C\int_{\mathcal{R}}\zeta^2\left|\vec{g}_z\cdot\nabla w\right|\left|\partial^2 w\right|^{2\beta}\\&+\frac{C}{\alpha}\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta}+C\int_{\mathcal{R}}\zeta^2\left|\left(\partial\vec{g}\right)\nabla\left(\partial w\right)\right|\left|\partial^2 w\right|^{2\beta-2}\\&+\alpha\int_{\mathcal{R}}\zeta^2\left|\left(\partial\mathbf{A}\right)\nabla\left(\partial^2 w\right)^{\beta}\right|^2+\alpha\int_{\mathcal{R}}\zeta^2\left|\vec{g}\cdot\nabla\left(\partial^2 w\right)^{\beta}\right|^2.\end{aligned}$$

Proof. By (5.9) in the appendix, we have

$$\begin{aligned}&\mathcal{L}\left(\partial^2 w\right)\\=\;&-\nabla^t\left(\partial^2\mathbf{A}\right)\nabla w+c_1\,\nabla^t\left(\partial\mathbf{A}\right)\nabla\left(\partial w\right)+f_z\left(\partial^2 w\right)\\&+\left(\partial^2 w\right)\vec{g}_z\cdot\nabla w+\vec{g}\cdot\nabla\left(\partial^2 w\right)+c_2\left(\partial\vec{g}\right)\nabla\left(\partial w\right).\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\int_{\mathcal{R}}\left(\zeta\mathcal{L}\partial^2 w\right)\left(\zeta\left(\partial^2 w\right)^{2\beta-1}\right)\right|\\\leq\;&\left|\int_{\mathcal{R}}\left(\nabla^t\left(\partial^2\mathbf{A}\right)\nabla w\right)\left(\zeta^2\left(\partial^2 w\right)^{2\beta-1}\right)\right|+C\left|\int_{\mathcal{R}}\left(\nabla^t\left(\partial\mathbf{A}\right)\nabla\left(\partial w\right)\right)\left(\zeta^2\left(\partial^2 w\right)^{2\beta-1}\right)\right|\\&+\left|\int_{\mathcal{R}}\zeta^2 f_z\left(\partial^2 w\right)^{2\beta}\right|+\left|\int_{\mathcal{R}}\zeta^2\left(\vec{g}_z\cdot\nabla w\right)\left(\partial^2 w\right)^{2\beta}\right|\\&+\left|\int_{\mathcal{R}}\left(\vec{g}\cdot\nabla\left(\partial^2 w\right)\right)\left(\zeta^2\left(\partial^2 w\right)^{2\beta-1}\right)\right|+C\left|\int_{\mathcal{R}}\left(\zeta\left(\partial\vec{g}\right)\nabla\left(\partial w\right)\right)\left(\zeta\left(\partial^2 w\right)^{2\beta-1}\right)\right|\\=\;&I+II+III+IV+V+VI.\end{aligned}$$

We proceed to estimate each one of these terms. Integrating by parts, we have

$$\begin{aligned}II\;=\;&C\left|\int_{\mathcal{R}}\left(\nabla\left(\partial w\right)\right)\left(\partial\mathbf{A}\right)\nabla\zeta^2\left(\partial^2 w\right)^{2\beta-1}\right|\\\leq\;&C\left|\int_{\mathcal{R}}\left(\nabla\left(\partial w\right)\right)\left(\partial\mathbf{A}\right)\left(\nabla\zeta^2\right)\left(\partial^2 w\right)^{2\beta-1}\right|+C\left|\int_{\mathcal{R}}\left(\nabla\left(\partial w\right)\right)\left(\partial\mathbf{A}\right)\zeta^2\nabla\left(\partial^2 w\right)^{2\beta-1}\right|\\\leq\;&C\int_{\mathcal{R}}\left|\left(\partial\mathbf{A}\right)\nabla\zeta\right|^2\left|\partial^2 w\right|^{2\beta}+\frac{C}{\alpha}\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta}+\alpha\int_{\mathcal{R}}\zeta^2\left|\left(\partial\mathbf{A}\right)\nabla\left(\partial^2 w\right)^{\beta}\right|^2.\end{aligned}$$

and

$$\begin{aligned}III\;&\leq\;C\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta}\\IV\;&\leq\;\int_{\mathcal{R}}\zeta^2\left|\vec{g}_z\cdot\nabla w\right|\left|\partial^2 w\right|^{2\beta}\\V\;&\leq\;\alpha\int_{\mathcal{R}}\zeta^2\left|\vec{g}\cdot\nabla\left(\partial^2 w\right)^{\beta}\right|^2+\frac{C}{\alpha}\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta}\\VI\;&\leq\;C\int_{\mathcal{R}}\zeta^2\left|\left(\partial\vec{g}\right)\nabla\left(\partial w\right)\right|\left|\partial^2 w\right|^{2\beta-2}+C\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta}.\end{aligned}$$

The lemma follows from assembling all these estimates.

The following “Caccioppoli”-type result for second derivatives of solutions, is the essence of the proof of Theorem 3.8.

Lemma 3.10. *Suppose that w is a smooth solution of (1.13) in $\mathcal{R}$. Then for $\beta \geq 1$, the k-gradient integral*

$$\int_{\mathcal{R}} \left|\zeta \nabla_k \left(\nabla^2 w\right)^{\beta}\right|^2 \tag{3.27}$$

is dominated by

$$\begin{aligned}&\mathcal{C}\left(A, B, \beta, \|\xi \nabla w\|_{\infty}\right)\left\{1+\int_{\mathcal{R}} \xi^2\left|\partial^2 w\right|^{2\beta-2}\right\}\\ &+\mathcal{C}\left(B, \beta, \|\xi \nabla w\|_{\infty}\right) \int_{\mathcal{R}}\left\{\zeta^2+\left|\nabla_k \zeta\right|^2\right\}\left|\partial^2 w\right|^{2\beta}.\end{aligned}$$

A crucial point is that the constants on the second term of the conclusion in the above lemma do not depend on A, so that in applying the one dimensional Poincaré inequality in the proof of Theorem 3.8, the product $R_1^2 \mathcal{C}_1$ can be made less than one for R_1 sufficiently small. This would be impossible if A^2 were present since $A \geq R_1^{-1}$ - recall (2.3).

Proof. The result follows from applying Lemma 2.1 to the functions $\partial^2 w$, to obtain

$$\begin{aligned}&\int_{\mathcal{R}}\left|\zeta \nabla_k\left(\partial^2 w\right)^{\beta}\right|^2 dx \\ \leq\ & C \beta\left|\int_{\mathcal{R}}\left(\zeta \mathcal{L} \partial^2 w\right)\left(\zeta\left(\partial^2 w\right)^{2\beta-1}\right)\right|+C \beta^2 \int_{\mathcal{R}}\left|\nabla_k \zeta\right|^2\left|\partial^2 w\right|^{2\beta}.\end{aligned} \tag{3.28}$$

By Lemma 3.9, we have

$$\begin{aligned}&\left|\int_{\mathcal{R}}\left(\zeta \mathcal{L} \partial^2 w\right)\left(\zeta\left(\partial^2 w\right)^{2\beta-1}\right)\right| \\ \leq\ & C\left|\int_{\mathcal{R}}(\nabla w)\left(\partial^2 \mathbf{A}\right) \nabla \zeta^2\left(\partial^2 w\right)^{2\beta-1}\right| \\ &+C \int_{\mathcal{R}}|(\partial \mathbf{A}) \nabla \zeta|^2\left|\partial^2 w\right|^{2\beta}+C \int_{\mathcal{R}} \zeta^2\left|\vec{g}_z \cdot \nabla w\right|\left|\partial^2 w\right|^{2\beta} \\ &+\frac{C}{\alpha} \int_{\mathcal{R}} \zeta^2\left|\partial^2 w\right|^{2\beta}+C \int_{\mathcal{R}} \zeta^2|(\partial \vec{g}) \nabla(\partial w)|\left|\partial^2 w\right|^{2\beta-2} \\ &+\alpha \int_{\mathcal{R}} \zeta^2\left|(\partial \mathbf{A}) \nabla\left(\partial^2 w\right)^{\beta}\right|^2+\alpha \int_{\mathcal{R}} \zeta^2\left|\vec{g} \cdot \nabla\left(\partial^2 w\right)^{\beta}\right|^2 .\end{aligned} \tag{3.29}$$

Integrating by parts the first term on the right, we get

$$\begin{aligned}&\left|\int_{\mathcal{R}}(\nabla w)\left(\partial^2 \mathbf{A}\right) \nabla \zeta^2\left(\partial^2 w\right)^{2\beta-1}\right| \\ \leq\ & \left|\int_{\mathcal{R}}(\nabla w)\left(\partial^2 \mathbf{A}\right)\left(\nabla \zeta^2\right)\left(\partial^2 w\right)^{2\beta-1}\right|+\left|\int_{\mathcal{R}}(\nabla w)\left(\partial^2 \mathbf{A}\right) \zeta^2 \nabla\left(\partial^2 w\right)^{2\beta-1}\right| \\ \leq\ & C \int_{\mathcal{R}}|\nabla w|^2\left|\left(\partial^2 \mathbf{A}\right) \nabla \zeta\right|^2\left|\partial^2 w\right|^{2\beta-2}+\frac{C}{\alpha} \int_{\mathcal{R}} \zeta^2\left|\partial^2 w\right|^{2\beta} \\ &+\left|\int_{\mathcal{R}}(\nabla w)\left(\partial^2 \mathbf{A}\right) \zeta^2 \nabla\left(\partial^2 w\right)^{2\beta-1}\right| .\end{aligned}$$

Replacing back into (3.29) yields

$$\begin{aligned}
&\left|\int_{\mathcal{R}}\left(\zeta\mathcal{L}\partial^2 w\right)\left(\zeta\left(\partial^2 w\right)^{2\beta-1}\right)\right| \\
\leq\ & C\int_{\mathcal{R}}|\nabla w|^2\left|\left(\partial^2\mathbf{A}\right)\nabla\zeta\right|^2\left|\partial^2 w\right|^{2\beta-2}+C\int_{\mathcal{R}}\left|\left(\partial\mathbf{A}\right)\nabla\zeta\right|^2\left|\partial^2 w\right|^{2\beta} \\
&+C\int_{\mathcal{R}}\zeta^2\left|\left(\partial\vec{g}\right)\nabla\left(\partial w\right)\right|\left|\partial^2 w\right|^{2\beta-2}+\left|\int_{\mathcal{R}}\zeta^2\left(\nabla w\right)\left(\partial^2\mathbf{A}\right)\nabla\left(\partial^2 w\right)^{2\beta-1}\right| \\
&+\int_{\mathcal{R}}\zeta^2\left|\vec{g}_z\cdot\nabla w\right|\left|\partial^2 w\right|^{2\beta}+\frac{C}{\alpha}\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta} \\
&+\alpha\int_{\mathcal{R}}\zeta^2\left|\left(\partial\mathbf{A}\right)\nabla\left(\partial^2 w\right)^{\beta}\right|^2+\alpha\int_{\mathcal{R}}\zeta^2\left|\vec{g}\cdot\nabla\left(\partial^2 w\right)^{\beta}\right|^2 \\
=\ & I+II+\cdots+VIII. \qquad (3.30)
\end{aligned}$$

From the chain rule, we have

$$\partial^2\mathbf{A}=\mathbf{A}_{\nu\nu}+2\left(\partial w\right)\mathbf{A}_{\nu z}+\left(\partial w\right)^2\mathbf{A}_{zz}+\left(\partial^2 w\right)\mathbf{A}_z \tag{3.31}$$

it follows that

$$\begin{aligned}
\left|\left(\partial\mathbf{A}\right)\nabla\zeta\right|^2 &\leq C\left|\mathbf{A}_\nu\nabla\zeta\right|^2+C\left|\partial w\right|^2\left|\mathbf{A}_z\nabla\zeta\right|^2 \\
\left|\left(\partial^2\mathbf{A}\right)\nabla\zeta\right|^2 &\leq CA^2\left(1+\left|\partial w\right|^2\right)^2+C\left|\left(\partial^2 w\right)\mathbf{A}_z\nabla\zeta\right|^2.
\end{aligned}$$

Hence, from (1.19)

$$I+II\leq\mathcal{C}\left(A,\left\|\xi\nabla w\right\|_\infty\right)\int_{\mathcal{R}}\xi^2\left|\partial^2 w\right|^{2\beta-2}+\mathcal{C}\left(B,\left\|\xi\nabla w\right\|_\infty\right)\int_{\mathcal{R}}\left|\nabla_k\zeta\right|^2\left|\partial^2 w\right|^{2\beta}. \tag{3.32}$$

Since $\left|\partial\vec{g}\right|\leq C\left|\nabla w\right|_\infty$, it easily follows that

$$III+V\leq\mathcal{C}\left(\left\|\xi\nabla w\right\|_\infty\right)\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta}. \tag{3.33}$$

Now, from (3.31), integration by parts and (1.19), we have

$$\begin{aligned}
IV\ \leq\ & \left|\int_{\mathcal{R}}\left(\partial^2 w\right)^{2\beta-1}\nabla\left\{\zeta^2\left(\nabla w\right)\left(\mathbf{A}_{\nu\nu}+2\left(\partial w\right)\mathbf{A}_{\nu z}+\left(\partial w\right)^2\mathbf{A}_{zz}\right)\right\}\right| \\
&+\left|\int_{\mathcal{R}}\zeta^2\left(\nabla w\right)\mathbf{A}_z\nabla\left(\partial^2 w\right)^{2\beta}\right| \\
\leq\ & \mathcal{C}\left(A,\left\|\xi\nabla w\right\|_\infty\right)\int_{\mathcal{R}}\xi^2\left|\partial^2 w\right|^{2\beta-2}+\frac{1}{\alpha}\mathcal{C}\left(\left\|\xi\nabla w\right\|_\infty\right)\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta} \qquad (3.34)\\
&+\alpha\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)^{\beta}\right|^2.
\end{aligned}$$

Finally, from (1.19) it follows that

$$VII+VIII\leq C\,\alpha\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)^{\beta}\right|^2 \tag{3.35}$$

Gathering the estimates (3.28) to (3.35) yields

$$\begin{aligned}
&\int_{\mathcal{R}}\left|\zeta\nabla_k\left(\partial^2 w\right)^\beta\right|^2 dx\\
\le\ & \mathcal{C}\left(A,\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}\xi^2\left|\partial^2 w\right|^{2\beta-2}+\mathcal{C}\left(B,\beta,\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}\left\{\zeta^2+\left|\nabla_k\zeta\right|^2\right\}\left|\partial^2 w\right|^{2\beta}\\
&+\frac{1}{\alpha}\mathcal{C}\left(\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}\left|\partial^2 w\right|^{2\beta}+C\,\alpha\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)^\beta\right|^2+\mathcal{C}
\end{aligned}\tag{3.36}$$

Using the one-dimensional Poincaré inequality (2.4), we have that for $\beta>1$,

$$\begin{aligned}
&\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta}\\
\le\ & CR_1^2\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)^\beta\right|^2+CR_1^2\int_{\mathcal{R}}\zeta_1^2\left|\partial^2 w\right|^{2\beta},
\end{aligned}\tag{3.37}$$

where we wrote $\zeta_1=\partial_1\zeta$. Completing the k-gradient on the second term, and replacing this inequality into the right of (3.36), we conclude

$$\begin{aligned}
&\int_{\mathcal{R}}\left|\zeta\nabla_k\left(\partial^2 w\right)^\beta\right|^2 dx\\
\le\ & \mathcal{C}\left(A,\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}\xi^2\left|\partial^2 w\right|^{2\beta-2}+\mathcal{C}\left(B,\beta,\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}\left\{\zeta^2+\left|\nabla_k\zeta\right|^2\right\}\left|\partial^2 w\right|^{2\beta}\\
&+\mathcal{C}+C\left(\alpha+\frac{R_1^2}{\alpha}\mathcal{C}\left(\|\xi\nabla w\|_\infty\right)\right)\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)^\beta\right|^2.
\end{aligned}$$

The lemma follows from applying this inequality to the right side of (3.28) and absorbing the last term into the left by taking α, R_1 small enough.

Proof of Theorem 3.8. From Poincaré inequality (3.37) and Lemma 3.10 we have

$$\begin{aligned}
&\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta}\\
\le\ & \mathcal{C}\left(A,B,\beta,\|\xi\nabla w\|_\infty\right)\left\{1+\int_{\mathcal{R}}\xi^2\left|\partial^2 w\right|^{2\beta-2}\right\}\\
&+R_1^2\,\mathcal{C}\left(B,\beta,\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}\left\{\zeta^2+\left|\nabla_k\zeta\right|^2\right\}\left|\partial^2 w\right|^{2\beta},
\end{aligned}\tag{3.38}$$

Now, by Lemma 2.12 we have

$$\int_{\mathcal{R}}\left|\nabla_k\zeta\right|^2\left|\partial^2 w\right|^{2\beta}\le\mathcal{C}\left(A\right)\left(\mathcal{K}^2+\frac{1}{\varepsilon}\right)\left(\int_{\mathcal{R}}\left|\xi\left(\partial^2 w\right)^\beta\right|^p\right)^{\frac{2}{p}}+\varepsilon\int_{\mathcal{R}}\left|\zeta\nabla_k\left(\partial^2 w\right)^\beta\right|^2\tag{3.39}$$

$\nu>0$ and $q>\max\left\{\frac{2}{1-\frac{\nu}{6}},n\left(1+\nu\right)\right\}$by Lemma 2.12, where $0<\varepsilon\le 1$ is arbitrary, $1<p<2$ and $\mathcal{K}=\left\|\varkappa\nabla\mathbf{A}\right\|_{L^q}$ with $q>\max\left\{\frac{2}{1-\frac{\nu}{6}},n\left(1+\nu\right)\right\}$, $\nu>0$, notice that since we can take q just depending on the dimension n, we have

$$\mathcal{K}\le\mathcal{C}\left(\|\xi\nabla w\|_\infty\right).$$

In turn, by Lemma 3.10 applied to the second term on the right-hand side of (3.39), and the above inequality, we have

$$\begin{aligned}\int_{\mathcal{R}} |\nabla_k\zeta|^2 \left|\partial^2 w\right|^{2\beta} &\leq \mathcal{C}\left(A,\beta,B,\|\xi\nabla w\|_\infty\right)\left(\int_{\mathcal{R}}\left|\xi\left(\partial^2 w\right)^\beta\right|^p\right)^{\frac{2}{p}}\\ &\quad+\mathcal{C}\left(A,B,\beta,\|\xi\nabla w\|_\infty\right)\left\{1+\int_{\mathcal{R}}\xi^2\left|\partial^2 w\right|^{2\beta-2}\right\}\end{aligned}$$

after we absorbed $\varepsilon\int_{\mathcal{R}}|\nabla_k\zeta|^2\left|\partial^2 w\right|^{2\beta}$ into the left-hand side, by taken ε small enough. Finally, replacing this inequality into the last term in the right side of (3.38), and absorbing terms into the left, we obtain

$$\begin{aligned}\int_{\mathcal{R}} \zeta^2 \left|\partial^2 w\right|^{2\beta} &\leq \mathcal{C}\left(A,\beta,B,\|\xi\nabla w\|_\infty\right)\left(\int_{\mathcal{R}}\left|\xi\left(\partial^2 w\right)^\beta\right|^p\right)^{\frac{2}{p}}\\ &\quad+\mathcal{C}\left(A,B,\beta,\|\xi\nabla w\|_\infty\right)\left\{1+\int_{\mathcal{R}}\xi^2\left|\partial^2 w\right|^{2\beta-2}\right\}.\end{aligned}\tag{3.40}$$

Since from Lemma (3.2) we have that $\nabla^2 u\in L^2_{\text{loc}}(\Omega)$, i.e. $\left\|\xi\nabla^2 u\right\|_0\leq\mathcal{C}\left(A,B,\|\xi\nabla w\|_\infty\right)$, we can take $\beta_1=\frac{2}{p}$ in the above inequality, and since then $2\beta_1-2<2$, we obtain

$$\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{\frac{4}{p}}\leq\mathcal{C}\left(A,\beta,B,\|\xi\nabla w\|_\infty\right).\tag{3.41}$$

Next, taking $\beta_2=\min\left\{\frac{2}{p}\beta_1,\beta_1+1\right\}$ in (3.40), (notice that both $p\beta_2$ and $2\beta_2-2$ are dominated by $2\beta_1=\frac{4}{p}$), and applying (3.41) yields

$$\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta_2}\leq\mathcal{C}\left(A,\beta,B,\|\xi\nabla w\|_\infty\right).$$

We can continue this iteration process to obtain

$$\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta_{m+1}}\leq\mathcal{C}\left(A,\beta,B,\|\xi\nabla w\|_\infty\right),\qquad m=1,2,\dots$$

where $\beta_{m+1}=\min\left\{\frac{2}{p}\beta_m,\beta_m+1\right\}$. Since $\beta_1=\frac{2}{p}$, it is easy to see that

$$\beta_m\geq\frac{2}{p}+m\left(\frac{2-p}{p}\right),$$

and therefore

$$\left\|\zeta\nabla^2 w\right\|_{L^q}\leq\mathcal{C}\left(\|\varkappa\nabla w\|_\infty,L\right),$$

for all $q>1$, provided R_1 is sufficiently small, depending on q. This finishes the proof oh the theorem.

3.3. **Proof of Theorem 1.4.** We use the phrase "*under control*" to mean bounded by an increasing function $\mathcal{C}=\mathcal{C}\left(\|\varkappa\nabla w\|_\infty,L,A,B\right)$, where $L\in\mathcal{P}_c(\Omega)$.

We will estimate the L^2_{loc} norm of $(\partial^m w)$ by the technique of the proof of Theorem 3.2. In this process, we apply Lemma 2.5 to $(\partial^m w)$, and we will use the expression (5.8) for $\mathcal{L}(\partial^m w)$, computed on the appendix . We will prove by induction on m, that the following holds

$$\nabla_k\left(\partial^{m+1}w\right) \in L^q_{\mathrm{loc}} \textit{ with control}, \quad 2\leq q<\infty, \quad q=q\left(R_1\right), \tag{3.42}$$

$$\nabla_k\left(\partial^{m+1}w\right) \in L^2_{\mathrm{loc}} \textit{ with control}, \tag{3.43}$$

$$\left(\partial w\right),\ldots,\left(\partial^{m}w\right) \in L^q_{\mathrm{loc}}, \textit{ with control}, \quad 2\leq q\leq\infty. \tag{3.44}$$

for all $m=1,2,\,3,\ldots$.

First, note that these conditions hold for $m=1$. Indeed, trivially we have $\left(\partial w\right)\in L^\infty_{\mathrm{loc}}$ *with control.* By Theorem 3.2 we have that (3.43) holds, and Lemma 3.10 implies (3.42). Now, by Lemma 2.5 we get

$$\begin{aligned}\left\|\nabla_k\eta\partial^{m+1}w\right\|_0^2 \;\leq\;& CA^6\kappa^2\left|\int_{\mathcal{R}}\left(\eta\mathcal{L}\partial^m w\right)\left(\zeta\partial^m w\right)\right| \\ &+C\left|\int_{\mathcal{R}}\left(\eta\partial\mathcal{L}\partial^m w\right)\left(\eta\partial^{m+1}w\right)\right| \\ &+C\kappa^2\left\|\eta\partial^{m+1}w\right\|_0^2+CA^6\kappa^2\left\|\xi\partial^m w\right\|_0^2,\end{aligned} \tag{3.45}$$

where $\kappa=\kappa\left(w\right)$ is as in Definition 2.4. i.e

$$\kappa\left(w\right)^2=\sup_{\det\left(\mathbf{A}\left(x,u\left(x\right)\right)\right)\neq0\neq\left|\xi\right|}\frac{\left|\left(\partial\tilde{\mathbf{A}}_w\right)\xi\right|^2}{\xi^t\mathbf{A}\left(x,w\left(x\right)\right)\xi}.$$

where $\tilde{\mathbf{A}}_w\left(x\right)=\mathbf{A}\left(x,w\left(x\right)\right)$. Since from (1.18) we have

$$\begin{aligned}\left|\left(\partial\tilde{\mathbf{A}}_w\right)\xi\right|^2 &\leq \left|\mathbf{A}_\nu\xi\right|^2+\left|\left(w_i\right)\mathbf{A}_z\xi\right|^2 \\ &\leq CB^2\left(1+\left\|\varkappa\nabla w\right\|_\infty^2\right)\xi^t\mathbf{A}\xi,\end{aligned}$$

then

$$\kappa\left(w\right)^2\leq CB^2\left(1+\left\|\varkappa\nabla w\right\|_\infty^2\right). \tag{3.46}$$

The third term in (3.45) can be controlled applying (3.26) and the one dimensional Poincaré inequality (2.4),

$$C\kappa^2\left\|\eta\partial^{m+1}w\right\|_0^2\leq CB^2\left(1+\left\|\varkappa\nabla w\right\|_\infty^2\right)R_1^2\left\|\nabla_k\eta\partial^{m+1}u\right\|_0^2,$$

and this can be absorbed into the left side for R_1 small enough. Hence,

$$\begin{aligned}\left\|\nabla_k\eta\partial^{m+1}w\right\|_0^2 \;\leq\;& CA^4\kappa^2\left|\int_{\mathcal{R}}\left(\eta\mathcal{L}\partial^m w\right)\left(\zeta\partial^m w\right)\right| \\ &+C\left|\int_{\mathcal{R}}\left(\eta\partial\mathcal{L}\partial^m w\right)\left(\eta\partial^{m+1}w\right)\right|+\mathcal{C}\left(A,\kappa,\left\|\xi\partial^m w\right\|_0\right).\end{aligned} \tag{3.47}$$

Now we assume that

$$\text{(3.42), (3.43) and (3.44) hold for } m=1,2,\ldots,M. \tag{3.48}$$

From (3.47), to prove that (3.48) holds for $M+1$, it is enough to show that

$$\begin{aligned}&\left|\int_{\mathcal{R}}\left(\eta\partial\mathcal{L}\partial^{M+1}w\right)\left(\eta\partial^{M+2}w\right)\right| \\ \leq\;& C\left(\alpha+R_1\left\|\zeta\nabla w\right\|_\infty+\frac{R_1}{\alpha}\right)\int_{\mathcal{R}}\left|\nabla_k\eta\left(\partial^{M+2}w\right)\right|^2+\mathcal{C}\left(A,B,\left\|\varkappa\nabla w\right\|_\infty\right).\end{aligned} \tag{3.49}$$

From (5.8) we have

$$\left|\int_{\mathcal{R}}\left(\eta\partial\mathcal{L}\partial^{M+1}w\right)\left(\eta\partial^{M+2}w\right)\right|\leq C\left(I+II+\cdots+IX\right), \tag{3.50}$$

where

(3.51)

$$\begin{aligned}
I &= \left|\int_{\mathcal{R}}\left(\eta\partial\nabla^{t}\left(\partial^{M+1}\mathbf{A}\right)\nabla w\right)\left(\eta\partial^{M+2}w\right)\right|,\\
II &= \left|\int_{\mathcal{R}}\left(\eta\partial\nabla^{t}\left(\partial\mathbf{A}\right)\nabla\left(\partial^{M}w\right)\right)\left(\eta\partial^{M+2}w\right)\right|,\\
III &= \left|\int_{\mathcal{R}}\left(\eta\partial f_{z}\left(\partial^{M+1}w\right)\right)\left(\eta\partial^{M+2}w\right)\right|,\\
IV &= \left|\int_{\mathcal{R}}\left(\eta\partial\left(\partial^{M+1}w\right)\vec{g}_{z}\cdot\nabla w\right)\left(\eta\partial^{M+2}w\right)\right|,\\
V &= \left|\int_{\mathcal{R}}\left(\eta\partial\vec{g}\cdot\nabla\left(\partial^{M+1}w\right)\right)\left(\eta\partial^{M+2}w\right)\right|,\\
VI &= \left|\int_{\mathcal{R}}\left(\eta\partial\left(\partial\vec{g}\right)\nabla\left(\partial^{M}w\right)\right)\left(\eta\partial^{M+2}w\right)\right|,\\
VII &= \delta_{M\geq2}\left|\int_{\mathcal{R}}\left(\eta\partial\left[c_{3}\nabla^{t}\left(\partial^{M}\mathbf{A}\right)\nabla\left(\partial w\right)+c_{5}\left(\partial^{2}\vec{g}\right)\nabla\left(\partial^{M-1}w\right)\right]\right)\left(\eta\partial^{M+2}w\right)\right|,\\
VIII &= \delta_{M\geq3}\left|\int_{\mathcal{R}}\left(\eta\partial\left[c_{4}\nabla^{t}\left(\partial^{2}\mathbf{A}\right)\nabla\left(\partial^{M-1}w\right)+c_{6}\left(\partial^{M}\vec{g}\right)\nabla\left(\partial w\right)\right]\right)\left(\eta\partial^{M+2}w\right)\right|,\\
IX &= \delta_{M\geq4}\left|\int_{\mathcal{R}}\left(\eta\partial\,\mathcal{P}_{M+3}^{\mathbf{A},\vec{g},f}\left(1,\partial w,\ldots,\partial^{M}w\right)\right)\left(\eta\partial^{M+2}w\right)\right|.
\end{aligned}$$

We claim that

$$\begin{aligned}
&\left|\int_{\mathcal{R}}\left(\eta\partial\mathcal{L}\partial^{M+1}w\right)\left(\eta\partial^{M+2}w\right)\right|\\
&\leq\left\{\alpha+\frac{R_{1}^{2}}{\alpha}\mathcal{C}\left(B,\left\|\xi\nabla w\right\|_{\infty}\right)\right\}\int_{\mathcal{R}}\eta^{2}\left|\nabla_{k}\left(\partial^{M+2}w\right)\right|^{2}+\mathcal{C}\left(A,B,\left\|\xi\nabla w\right\|_{\infty}\right).
\end{aligned}\tag{3.52}$$

Now, since from Lemma 2.3 and (3.48) it follows that

$$\int_{\mathcal{R}}\left|\nabla_{k}\eta\right|^{2}\left|\partial^{M+2}w\right|^{2}\leq C\int_{\mathcal{R}}\left|\nabla\eta\right|^{2}\left|\nabla_{k}\left(\partial^{M+1}w\right)\right|^{2}\leq\mathcal{C}\left(A,B,\left\|\xi\nabla w\right\|_{\infty}\right),$$

then

$$\begin{aligned}
\left\|\eta\nabla_{k}\partial^{M+2}w\right\|_{0}^{2} &\leq \int_{\mathcal{R}}\left|\nabla_{k}\eta\partial^{M+2}w-\left(\nabla_{k}\eta\right)\partial^{M+2}w\right|^{2}\\
&\leq C\int_{\mathcal{R}}\left|\nabla_{k}\eta\left(\partial^{M+2}w\right)\right|^{2}+C\int_{\mathcal{R}}\left|\nabla_{k}\eta\right|^{2}\left|\partial^{M+2}w\right|^{2}\\
&\leq \mathcal{C}\left(A,B,\left\|\xi\nabla w\right\|_{\infty}\right)+\int_{\mathcal{R}}\eta^{2}\left|\nabla_{k}\left(\partial^{M+2}w\right)\right|^{2}.
\end{aligned}$$

Hence our claim (3.52) implies (3.49) and this proves the theorem. Inequality (3.52) is obtained by bounding each term in (3.51) separately. We will now just sketch the proof of (3.52) treating only some representative terms.

Operating the derivative on the left, we have

$$I\leq\left|\int_{\mathcal{R}}\left(\eta\nabla^{t}\left(\partial^{M+2}\mathbf{A}\right)\nabla w\right)\left(\eta\partial^{M+2}w\right)\right|+\left|\int_{\mathcal{R}}\left(\eta\nabla^{t}\left(\partial^{M+1}\mathbf{A}\right)\nabla\left(\partial w\right)\right)\left(\eta\partial^{M+2}w\right)\right|$$

We write

$$\begin{aligned}
\left(\partial^{M+2}\mathbf{A}\right) &= \left(\partial^{M+2}w\right)\mathbf{A}_z + \left(\partial^{M+1}w\right)\,\mathcal{P}_1^{\mathbf{A}}\left(1,\partial w\right) + \mathcal{P}_{M+2}^{\mathbf{A}}\left(1,\partial w,\dots,\partial^M w\right)\\
\left(\partial^{M+1}\mathbf{A}\right) &= \left(\partial^{M+1}w\right)\mathbf{A}_z + \mathcal{P}_{M+1}^{\mathbf{A}}\left(1,\partial w,\dots,\partial^M w\right),
\end{aligned}$$

and replacing in the above expression, we obtain

$$\begin{aligned}
I &\leq \left|\int_{\mathcal{R}}\left(\eta\nabla^t\left(\partial^{M+2}w\right)\mathbf{A}_z\nabla w\right)\left(\eta\partial^{M+2}w\right)\right|\\
&\quad+\left|\int_{\mathcal{R}}\left(\eta\nabla^t\left(\partial^{M+1}w\right)\,\mathcal{P}_1^{\mathbf{A}}\left(1,\partial w\right)\nabla w\right)\left(\eta\partial^{M+2}w\right)\right|\\
&\quad+\left|\int_{\mathcal{R}}\left(\eta\nabla^t\mathcal{P}_{M+2}^{\mathbf{A}}\left(1,\dots,\partial^M w\right)\nabla w\right)\left(\eta\partial^{M+2}w\right)\right|\\
&\quad+\left|\int_{\mathcal{R}}\left(\eta\nabla^t\left(\partial^{M+1}w\right)\mathbf{A}_z\nabla\left(\partial w\right)\right)\left(\eta\partial^{M+2}w\right)\right|\\
&\quad+\left|\int_{\mathcal{R}}\left(\eta\nabla^t\mathcal{P}_{M+1}^{\mathbf{A}}\left(1,\dots,\partial^M w\right)\nabla\left(\partial w\right)\right)\left(\eta\partial^{M+2}w\right)\right|\\
&= X+XI+XII+XIII+XIV.
\end{aligned}\tag{3.53}$$

Integrating by parts and using (1.18) we have

$$\begin{aligned}
X &\leq \left|\int_{\mathcal{R}}\left(\nabla w\right)^t\mathbf{A}_z\left(\nabla\eta^2\right)\left(\partial^{M+2}w\right)^2\right| + \left|\int_{\mathcal{R}}\eta^2\left(\partial^{M+2}w\right)\left(\nabla w\right)^t\mathbf{A}_z\left(\nabla\partial^{M+2}w\right)\right|\\
&\leq \frac{1}{\alpha}\mathcal{C}\left(B,\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}\left(\eta^2+\left|\nabla_k\eta\right|^2\right)\left|\partial^{M+2}w\right|^2 + \alpha\int_{\mathcal{R}}\eta^2\left|\nabla_k\left(\partial^{M+2}w\right)\right|^2.
\end{aligned}$$

Applying this to the inequality above for X, and from Poincaré inequality we get

$$X\leq\left\{\alpha+\frac{R_1^2}{\alpha}\mathcal{C}\left(B,\|\xi\nabla w\|_\infty\right)\right\}\int_{\mathcal{R}}\eta^2\left|\nabla_k\left(\partial^{M+2}w\right)\right|^2+\mathcal{C}\left(A,B,\|\xi\nabla w\|_\infty\right).$$

Writing

$$\nabla^t\left(\partial^{M+1}w\right)\,\mathcal{P}_1^{\mathbf{A}}\left(1,\partial w\right)\nabla w = \left(\partial^{M+2}w\right)\,\mathcal{P}_2^{\mathbf{A}}\left(1,\partial w\right)+\left(\partial^{M+1}w\right)\,\mathcal{P}_3^{\mathbf{A}}\left(1,\partial w,\partial^2 w\right),$$

and replacing in XI yields

$$\begin{aligned}
XI &\leq \mathcal{C}\left(\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}\eta^2\left|\partial^{M+2}w\right|^2+\mathcal{C}\left(\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}\eta^2\left|\left(\partial^{M+1}w\right)\right|^2\ \left|\partial^2 w\right|^2\\
&\leq \mathcal{C}\left(\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}\eta^2\left|\partial^{M+2}w\right|^2+\mathcal{C}\left(\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}\eta^2\ \left|\partial^2 w\right|^4\\
&\quad+\mathcal{C}\left(\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}\eta^2\left|\left(\partial^{M+1}w\right)\right|^4\\
&\leq R_1^2\mathcal{C}\left(\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}\eta^2\left|\nabla_k\left(\partial^{M+2}w\right)\right|^2+\mathcal{C}\left(A,B,\|\xi\nabla w\|_\infty\right).
\end{aligned}$$

where in the last inequality we applied (3.48). The other terms in (3.53) are handled similarly to give

$$I\leq\left\{\alpha+\frac{R_1^2}{\alpha}\mathcal{C}\left(B,\|\xi\nabla w\|_\infty\right)\right\}\int_{\mathcal{R}}\eta^2\left|\nabla_k\left(\partial^{M+2}w\right)\right|^2+\mathcal{C}\left(A,B,\|\xi\nabla w\|_\infty\right).$$

Operating the derivative on the left, using (1.20) and the Poincaré inequality we have

$$\begin{aligned} V &\leq \left|\int_{\mathcal{R}} \left(\eta\left(\partial\vec{g}\right)\cdot\nabla\left(\partial^{M+1}w\right)\right)\left(\eta\partial^{M+2}w\right)\right| + \left|\int_{\mathcal{R}} \left(\eta\vec{g}\cdot\nabla\left(\partial^{M+2}w\right)\right)\left(\eta\partial^{M+2}w\right)\right| \\ &\leq \frac{1}{\alpha}\mathcal{C}\left(B,\left\|\xi\nabla w\right\|_{\infty}\right)\int_{\mathcal{R}}\eta^2\left|\partial^{M+2}w\right|^2 + \alpha\int_{\mathcal{R}}\eta^2\left|\nabla_k\left(\partial^{M+2}w\right)\right|^2 \\ &\leq \left\{\alpha + \frac{R_1^2}{\alpha}\mathcal{C}\left(B,\left\|\xi\nabla w\right\|_{\infty}\right)\right\}\int_{\mathcal{R}}\eta^2\left|\nabla_k\left(\partial^{M+2}w\right)\right|^2. \end{aligned}$$

Writing

$$\partial\mathcal{P}_{M+3}^{\mathbf{A},\vec{g},f}\left(1,\partial w,\ldots,\partial^M w\right) = \mathcal{P}_{M+4}^{\mathbf{A},\vec{g},f}\left(1,\partial w,\ldots,\partial^{M+1}w\right),$$

and replacing in IX, we get

$$IX \leq \delta_{M\geq 4}\left|\int_{\mathcal{R}}\left(\eta\mathcal{P}_{M+4}^{\mathbf{A},\vec{g},f}\left(1,\partial w,\ldots,\partial^{M+1}w\right)\right)\left(\eta\partial^{M+2}w\right)\right|.$$

Since the above expression is non trivial only for $M \geq 4$, and since the leading terms of $\mathcal{P}_{M+4}^{\mathbf{A},\vec{g},f}\left(1,\partial w,\ldots,\partial^{M+1}w\right)$ are of the form $\left(\partial^{M+1}w\right)\mathcal{P}_{3}^{\mathbf{A},\vec{g},f}\left(1,\partial w,\partial^2 w,\partial^3 w\right)$, from (3.48) we have that

$$\left\|\eta\mathcal{P}_{M+4}^{\mathbf{A},\vec{g},f}\right\|_0 \leq \mathcal{C}\left(A,B,\left\|\xi\nabla w\right\|_{\infty}\right).$$

Hence IX is under control. The other terms in (3.51) are handled in a similar fashion to finish the proof of our claim (3.52). As we already noted, this implies that (3.49) holds, and therefore (3.48) holds for $m = M+1$. By induction on m, we have that (3.48) holds for all $m \geq 1$. From the Sobolev embedding theorem we then conclude

$$\nabla^m w \in L^{\infty},\ \textit{with control}, \quad m = 1,2,3,\ldots,$$

as wanted.

4. Hypoellipticity

In this section we prove Theorem 1.8, which is a hypoellipticity result for quasilinear operators in divergence form $\mathcal{L} = \nabla^t\mathbf{A}\nabla$, where $\mathbf{A}$ satisfies Hypotheses (1.1) and (1.7).

We will need the following pointwise a-priori estimate for solutions of (1.13),

$$\sqrt{k^*\left(x,w\right)}\sqrt{k^i\left(x,w\right)}\left|\partial_i w\right| \leq C, \quad i = 1,\ldots,n, \tag{4.1}$$

where $k^* = \min_i\left\{k^i\right\}$, (see Lemma 5.3 in the appendix). Note that from (1.23) and (4.1) we have

$$\begin{aligned} \left|\left(\partial w\right)\mathbf{A}^z\nabla w\right| &\leq CB\left|k^*\left(\partial w\right)\right|\left|\nabla_k w\right| \\ &\leq CB\left|\nabla_k w\right|. \end{aligned} \tag{4.2}$$

Moreover, since ϱ_i is supported where $k^j \geq c_1 = c_1\left(i,R_1\right)$ for $i \neq j$, we have

$$\varrho_i k^i\left(x,w\right) \leq C\,\varrho_i k^*\left(x,w\right) \leq C\,\varrho_i\sqrt{k^*\left(x,w\right)}\sqrt{k^j\left(x,w\right)}, \qquad i,j = 1,\ldots,n,$$

hence, from (4.1) we obtain

$$\varrho_i k^i\left(x,w\right)\left|\partial_j w\right| \leq C, \quad i,j = 1,\ldots,n. \tag{4.3}$$

As a consequence of (4.3) and Theorem 1.4, we actually obtain immediate a-priori control of any higher order derivatives of the solution w on the "elliptic domain" of the coefficients, more precisely, for any power β and multi-index α, there exists $C = C(c_1, B^*, \alpha, \beta)$ such that

$$\varrho_i \left|D^\alpha w\right|^\beta \leq C. \tag{4.4}$$

Indeed, from Theorem 1.4 we have that for any compact sets $\mathcal{G}$ and $\tilde{\mathcal{G}}$ with $\mathcal{G} \subset\subset \tilde{\mathcal{G}} \subset\subset \Gamma$ and multi-index α

$$\left\|D^\alpha w\right\|_{L^\infty(\mathcal{G})} \leq \mathcal{C}_\alpha\left(\left\|\nabla w\right\|_{L^\infty(\tilde{\mathcal{G}})}, \mathcal{G}\right), \quad |\alpha| \geq 0,$$

whenever $(x, w(x)) \in \tilde{\mathcal{G}}$ for every x in the projection of $\tilde{\mathcal{G}}$ onto Ω. Taking $\tilde{\mathcal{G}}$ as the support of ϱ_i, we have

$$\left\|\nabla w\right\|_{L^\infty(\tilde{\mathcal{G}})} \leq \frac{C}{\inf_{\tilde{\mathcal{G}}} k^*} \leq C^* = C^*\left(\vec{k}, R_1\right) < \infty.$$

Hence,

$$\left\|D^\alpha w\right\|_{L^\infty(\mathcal{G})} \leq \mathcal{C}_\alpha\left(\left\|\varkappa \nabla w\right\|_\infty, \mathcal{G}\right) = \mathcal{C}_\alpha\left(\tilde{\mathcal{G}}\right),$$

as wanted. Note that from the proof of Theorem 1.4, it can be seen that $\mathcal{C}_\alpha\left(\left\|\varkappa \nabla w\right\|_\infty, \mathcal{G}\right)$ is only polynomial on $\left\|\varkappa \nabla w\right\|_\infty$.

4.1. **An a priori estimate for quasilinear equations.**

Theorem 4.1. *Let w be a smooth solution of (1.13) in Ω, where $\mathbf{A}$, $f(x,z)$, $\vec{g}(x,z)$, $\vec{k}(x,z)$, are C^∞ functions defined in a domain $\Omega \times \{0\} \subset \Gamma \subset \Omega \times \mathbb{R}$, where $\mathbf{A}$ satisfies (1.18); $\vec{k}(x,z)$ satisfies Hypotheses 1.1 and 1.7, and $\vec{g}$ satisfies (1.20). Then, for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and compact $L \subset \Omega$, there exists a constant $\mathcal{C}_\alpha\left(\left\|w\right\|_{L^\infty(L)}, \mathbf{A}\right) > 0$ such that*

$$\left\|\zeta D^\alpha w\right\|_{L^\infty(\mathcal{G})} \leq \mathcal{C}_\alpha\left(\left\|\varkappa w\right\|_{L^\infty(\tilde{\mathcal{G}})}, \mathbf{A}\right). \tag{4.5}$$

For convenience, we will say that a function f is *under special control* (u.s.c.), if it satisfies a bound of the form (4.5), that is

$$\left\|\zeta f\right\|_{L^\infty(\mathcal{G})} \leq \mathcal{C}\left(\left\|\varkappa w\right\|_{L^\infty(\tilde{\mathcal{G}})}, \mathbf{A}\right).$$

The theorem is a consequence of the extra integrability of ∇w, which follows from the extra hypotheses on the coefficients, (1.7). We establish this fact first on the following lemma.

Lemma 4.2. *Let w be a smooth solution of (1.13) in Ω, and assume that $\mathbf{A}$, $f(x,z)$, $\vec{g}(x,z)$ and $\vec{k}(x,z)$ are as in Theorem (4.1). Then $\nabla w \in L^\beta_{\text{loc}}(\Omega)$, for all $1 \leq \beta < \infty$. Moreover,*

$$\int_{\mathcal{R}} \left|\zeta \nabla_k (\partial w)^\beta\right|^2 \leq \mathcal{C}\left(\mathbf{A}, A, B, B^*, \beta, \left\|w\right\|_{L^\infty(L)}\right).$$

Proof. From Lemma 2.1, we have that for all $\beta \geq 1$

$$\begin{aligned}\int_{\mathcal{R}} \left|\zeta \nabla_k (\partial w)^\beta\right|^2 \leq{}& C\,\beta \left|\int_{\mathcal{R}} \left(\zeta \mathcal{L}(\partial w)\right)\left(\zeta (\partial w)^{2\beta-1}\right)\right| \\ &+ C \int_{\mathcal{R}} \left|\nabla_k \zeta\right|^2 \left|(\partial w)^\beta\right|^2.\end{aligned} \tag{4.6}$$

From (3.3), the equality $\partial \mathbf{A} = \mathbf{A}_\nu + (\partial w)\,\mathbf{A}_z$, and integration by parts, we have

$$\begin{aligned} & \left|\int_{\mathcal{R}} (\zeta \mathcal{L}(\partial w)) \left(\zeta (\partial w)^{2\beta-1}\right)\right| \\ \leq\ & \left|\int_{\mathcal{R}} ((\mathbf{A}_\nu + (\partial w)\,\mathbf{A}_z)\nabla w)\left(\nabla \zeta^2 (\partial w)^{2\beta-1}\right)\right| \\ & + \left|\int_{\mathcal{R}} \left((\partial f) + \sum_{i=1}^n (\partial g^i)\, w_i\right)\left(\zeta^2 (\partial w)^{2\beta-1}\right)\right| \\ & + \left|\sum_{i=1}^n \int_{\mathcal{R}} \left(\zeta g^i (\partial_i \partial w)\right)\left(\zeta (\partial w)^{2\beta-1}\right)\right| \\ =\ & I + II + III. \end{aligned} \tag{4.7}$$

Writing

$$\left(\nabla \zeta^2 (\partial w)^{2\beta-1}\right) = \frac{2\beta-1}{\beta}\zeta^2 (\partial w)^{\beta-1}\left(\nabla (\partial w)^\beta\right) + \left(\nabla \zeta^2\right)(\partial w)^{2\beta-1}$$

we have

$$\begin{aligned} I \ \leq\ & C\left|\int_{\mathcal{R}} \zeta^2 (\partial w)^{\beta-1} (\nabla w)^t \mathbf{A}^\nu \nabla (\partial w)^\beta\right| \\ & + C\left|\int_{\mathcal{R}} \zeta^2 (\partial w)^{\beta} (\nabla w)^t \mathbf{A}^z \nabla (\partial w)^\beta\right| \\ & + \left|\int_{\mathcal{R}} (\partial w)^{2\beta-1} (\nabla w)^t \mathbf{A}^\nu \nabla \zeta^2\right| \\ & + \left|\int_{\mathcal{R}} (\partial w)^{2\beta} (\nabla w)^t \mathbf{A}^z \nabla \zeta^2\right|. \end{aligned} \tag{4.8}$$

From (1.23), (1.7) and (4.1), we have

$$\begin{aligned} |\nabla w|^2 \left|\mathbf{A}^z \nabla (\partial w)^\beta\right|^2 &\leq CB^2 \left|\nabla_k (\partial w)^\beta\right|^2 \\ & \text{and} \\ |\nabla w|^2 \left|\mathbf{A}^z \nabla \zeta\right|^2 &\leq CB^2 \left|\nabla_k \zeta\right|^2. \end{aligned}$$

Applying these inequalities to (4.8) we obtain

$$\begin{aligned} I \ \leq\ & \frac{CB^2}{\alpha}\int_{\mathcal{R}} \zeta^2 |\partial w|^{2\beta} + C\alpha \int_{\mathcal{R}} \zeta^2 \left|\nabla_k (\partial w)^\beta\right|^2 \\ & + C\int_{\mathcal{R}} |\partial w|^{2\beta} |\nabla_k \zeta|^2 + \mathcal{C}. \end{aligned} \tag{4.9}$$

Now, by (1.20) we have

$$\left|\sum_{i=1}^n g^i (\partial_i \partial w)(\partial w)^{2\beta-1}\right| \leq CB\left|\nabla_k (\partial w)^\beta\right| |\partial w|^\beta$$

hence

$$III \leq \frac{C}{\alpha}\int_{\mathcal{R}} \zeta^2 |\partial w|^{2\beta} + \alpha \int_{\mathcal{R}} \zeta^2 \left|\nabla_k (\partial w)^\beta\right|^2. \tag{4.10}$$

Finally, since from (1.25) and (4.1) we have

$$\begin{aligned}\left|\sum_{i=1}^{n}\left(\partial g^{i}\right)w_{i}\left(\partial w\right)^{2\beta-1}\right| &= \left|\sum_{i=1}^{n}g_{\nu}^{i}w_{i}\left(\partial w\right)^{2\beta-1}+\sum_{i=1}^{n}g_{z}^{i}w_{i}\left(\partial w\right)^{2\beta}\right|\\ &\leq CB\left|\partial w\right|^{2\beta}+CB\sum_{i=1}^{n}\left(k^{i}k^{*}\right)^{\frac{1}{2}}\left|w_{i}\right|\left|\partial w\right|^{2\beta}\\ &\leq CB\left|\partial w\right|^{2\beta}.\end{aligned}$$

Applying this to II yields

$$II\leq CB\int_{\mathcal{R}}\zeta^{2}\left|\partial w\right|^{2\beta}+\mathcal{C}. \tag{4.11}$$

Then ,replacing (4.9), (4.10) and (4.11) on the right of (4.7), and then using (4.6), we obtain

$$\begin{aligned}\int_{\mathcal{R}}\left|\zeta\nabla_{k}\left(\partial w\right)^{\beta}\right|^{2} &\leq \frac{CB^{2}}{\alpha}\int_{\mathcal{R}}\zeta^{2}\left|\partial w\right|^{2\beta}+C\alpha\int_{\mathcal{R}}\zeta^{2}\left|\nabla_{k}\left(\partial w\right)^{\beta}\right|^{2}\\ &\quad+C\int_{\mathcal{R}}\left|\partial w\right|^{2\beta}\left|\nabla_{k}\zeta\right|^{2}+\mathcal{C}.\end{aligned}$$

Applying the one dimensional Poincaré inequality to the first sum on the right and completing the k-gradient, we obtain

$$\int_{\mathcal{R}}\left|\zeta\nabla_{k}\left(\partial w\right)^{\beta}\right|^{2}\leq C\left(\alpha+R_{1}^{2}B^{*^{2}}\right)\int_{\mathcal{R}}\zeta^{2}\left|\nabla_{k}\left(\partial w\right)^{\beta}\right|^{2}+C\int_{\mathcal{R}}\left|\partial w\right|^{2\beta}\left|\nabla_{k}\zeta\right|^{2}+\mathcal{C}.$$

We absorb into the left the first term (given that α and R_1 are small enough), to obtain

$$\int_{\mathcal{R}}\left|\zeta\nabla_{k}\left(\partial w\right)^{\beta}\right|^{2}\leq\mathcal{C}+C\int_{\mathcal{R}}\left|\nabla_{k}\zeta\right|^{2}\left|\left(\partial w\right)^{\beta}\right|^{2}.$$

The second term on the right is u.s.c. because of (4.3). This finishes the proof of the lemma.

We are now ready to proceed with:

Proof of Theorem 4.1. By Theorem (1.4), we only need to show that ∇w is *under special control*, that is

$$\left\|\zeta\nabla w\right\|_{\infty}\leq\mathcal{C}_{\alpha}\left(\left\|\varkappa w\right\|_{\infty},\vec{k}\right).$$

By the Sobolev imbedding theorem, to prove this it is enough to show that

$$\left\|\zeta\partial^{2}w\right\|_{L^{\beta}}\leq\mathcal{C}_{\beta}\left(\left\|\varkappa w\right\|_{L^{\infty}},\vec{k}\right)\left\|\varkappa w\right\|_{\infty}, \tag{4.12}$$

for some $\beta>n$.

Applying Lemma 2.11 with $u=\partial^2 w$ and then applying Lemma 2.10 with $v=\left(\partial^{2}w\right)^{\beta}$, we obtain

$$\begin{aligned}&\int_{\mathcal{R}}\zeta^{2}\left|\nabla_{k}\left(\partial^{2}w\right)^{\beta}\right|^{2}\\ \leq\ &C\,\beta\left|\int_{\mathcal{R}}\left(\zeta\mathcal{L}\partial^{2}w\right)\left(\zeta\left(\partial^{2}w\right)^{2\beta-1}\right)\right|\\ &+\mathcal{C}\left(A,B,\alpha\right)\left\|\varkappa\nabla\mathbf{A}\right\|_{L^{q}}^{2}\left(\int_{\mathcal{R}}\left|\varkappa\left(\partial^{2}w\right)^{\beta}\right|^{p}\right)^{\frac{2}{p}},\end{aligned} \tag{4.13}$$

for all $\beta \geq 1$, $\nu > 0$, $q > (1+\nu)\,n$, and where $p = p\,(q,\nu,n)$, $1 < p < 2$. From Lemma 3.9, we have

$$\begin{aligned}
&\left|\int_{\mathcal{R}} \left(\zeta \mathcal{L} \partial^2 w\right) \left(\zeta \left(\partial^2 w\right)^{2\beta-1}\right)\right| \\
\leq\ & C \left|\int_{\mathcal{R}} \left(\nabla^t w\right) \left(\partial^2 \mathbf{A}\right) \nabla \zeta^2 \left(\partial^2 w\right)^{2\beta-1}\right| \\
& + C \int_{\mathcal{R}} \left|\left(\partial \mathbf{A}\right) \nabla \zeta\right|^2 \left|\partial^2 w\right|^{2\beta} + C \int_{\mathcal{R}} \zeta^2 \left|\vec{g}_z \cdot \nabla w\right| \left|\partial^2 w\right|^{2\beta} \\
& + \frac{C}{\alpha} \int_{\mathcal{R}} \zeta^2 \left|\partial^2 w\right|^{2\beta} + C \int_{\mathcal{R}} \zeta^2 \left|\left(\partial \vec{g}\right) \nabla \left(\partial w\right)\right| \left|\partial^2 w\right|^{2\beta-2} \\
& + \alpha \int_{\mathcal{R}} \zeta^2 \left|\left(\partial \mathbf{A}\right) \nabla \left(\partial^2 w\right)^{\beta}\right|^2 + \alpha \int_{\mathcal{R}} \zeta^2 \left|\vec{g} \cdot \nabla \left(\partial^2 w\right)^{\beta}\right|^2 \\
=\ & I + II + \cdots + VII. \qquad (4.14)
\end{aligned}$$

Writing

$$\left(\partial^2 \mathbf{A}\right) = \partial \left(\mathbf{A}_\nu + w_\nu \mathbf{A}_z\right) = \mathbf{A}_{\nu\nu} + 2 \left(\partial w\right) \mathbf{A}_{\nu z} + \left(\partial w\right)^2 \mathbf{A}_{zz} + \left(\partial^2 w\right) \mathbf{A}_z$$

we have that I is bounded by

$$\begin{aligned}
& C \left|\int_{\mathcal{R}} \left(\nabla w\right)^t \mathbf{A}_{\nu\nu} \nabla \zeta^2 \left(\partial^2 w\right)^{2\beta-1}\right| + C \left|\int_{\mathcal{R}} \left(\nabla w\right)^t \left(\partial w\right) \mathbf{A}_{\nu z} \nabla \zeta^2 \left(\partial^2 w\right)^{2\beta-1}\right| \\
& + C \left|\int_{\mathcal{R}} \left(\nabla w\right)^t \left(\partial w\right)^2 \mathbf{A}_{zz} \nabla \zeta^2 \left(\partial^2 w\right)^{2\beta-1}\right| + C \left|\int_{\mathcal{R}} \left(\nabla w\right)^t \mathbf{A}_z \nabla \zeta^2 \left(\partial^2 w\right)^{2\beta}\right| \\
(4.15)\ =\ & VIII + IX + X + XI.
\end{aligned}$$

Integrating by parts and using Hölder inequality, we obtain

$$\begin{aligned}
VIII &= C \left|\int_{\mathcal{R}} \left(\nabla^t \mathbf{A}_{\nu\nu} \nabla w\right) \zeta^2 \left(\partial^2 w\right)^{2\beta-1}\right| \\
&\leq C \int_{\mathcal{R}} \zeta^2 \left|\partial^2 w\right|^{2\beta} + C \int_{\mathcal{R}} \left(\left|\nabla w\right|^2 + \left|\nabla w\right|\right) \zeta^2 \left|\partial^2 w\right|^{2\beta-1} \\
&\leq C \int_{\mathcal{R}} \zeta^2 \left|\partial^2 w\right|^{2\beta} + C \int_{\mathcal{R}} \xi^2 \left|\nabla w\right|^{4\beta} + C. \qquad (4.16)
\end{aligned}$$

Similarly,

$$IX + X \leq C \int_{\mathcal{R}} \zeta^2 \left|\partial^2 w\right|^{2\beta} + C \int_{\mathcal{R}} \xi^2 \left|\nabla w\right|^{6\beta} + C. \tag{4.17}$$

To treat XI, we apply (1.23) to obtain

$$\begin{aligned}
XI &= C \left|\int_{\mathcal{R}} \left(\nabla w\right)^t \zeta \left(\partial^2 w\right)^{\beta} \mathbf{A}_z \nabla \zeta \left(\partial^2 w\right)^{\beta}\right| \\
&\leq \frac{C}{\alpha} \int_{\mathcal{R}} \zeta^2 \left|\partial^2 w\right|^{2\beta} + \alpha \int_{\mathcal{R}} \left|\nabla w\right|^2 \left|\mathbf{A}_z \nabla \zeta \left(\partial^2 w\right)^{\beta}\right|^2 \\
&\leq \frac{C}{\alpha} \int_{\mathcal{R}} \zeta^2 \left|\partial^2 w\right|^{2\beta} + \alpha C B^2 \int_{\mathcal{R}} \left|k^*\left(x,w\right) \nabla w\right|^2 \left|\nabla_k \zeta \left(\partial^2 w\right)^{\beta}\right|^2.
\end{aligned}$$

Since by (4.1) we have $\left|k^*\left(x,w\right) \nabla w\right| \leq C$, we get

$$XI \leq \frac{CB^2}{\alpha} \int_{\mathcal{R}} \zeta^2 \left|\partial^2 w\right|^{2\beta} + \alpha \int_{\mathcal{R}} \left|\nabla_k \zeta\right|^2 \left|\partial^2 w\right|^{2\beta} + \alpha \int_{\mathcal{R}} \zeta^2 \left|\nabla_k \left(\partial^2 w\right)^{\beta}\right|^2. \tag{4.18}$$

Gathering the estimates (4.15) to (4.18), we have

$$\begin{aligned} I &\leq C\int_{\mathcal{R}}\xi^2\left|\nabla w\right|^{6\beta}+C\int_{\mathcal{R}}\left|\nabla_k\zeta\right|^2\left|\partial^2 w\right|^{2\beta} \\ &\quad +\frac{CB^2}{\alpha}\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta}+\alpha\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)^{\beta}\right|^2+C. \end{aligned}$$

Using similar techniques to treat the other terms in (4.14), (with (1.23) replaced by (1.25), (1.20) when treating III, VII, respectively), we obtain

$$\begin{aligned} \left|\int_{\mathcal{R}}\left(\zeta\mathcal{L}\partial^2 w\right)\left(\zeta\left(\partial^2 w\right)^{2\beta-1}\right)\right| &\leq C\int_{\mathcal{R}}\xi^2\left|\nabla w\right|^{6\beta}+C\int_{\mathcal{R}}\left|\nabla_k\zeta\right|^2\left|\partial^2 w\right|^{2\beta} \\ &\quad +\left(\frac{CB^2}{\alpha}R_1^2+\alpha\right)\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)^{\beta}\right|^2+C, \end{aligned} \tag{4.19}$$

where we applied Poincaré inequality to $\int_{\mathcal{R}}\zeta^2\left|\partial^2 w\right|^{2\beta}$ as usual to bound it by

$$R_1^2\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)^{\beta}\right|^2+R_1^2\int_{\mathcal{R}}\left|\nabla_k\zeta\right|^2\left|\partial^2 w\right|^{2\beta}.$$

Since $\nabla_k\zeta$ is supported where $k^*\geq c=c\left(\vec{k},R_1\right)>0$, from (4.4) we have that $\int_{\mathcal{R}}\left|\nabla_k\zeta\right|^2\left|\partial^2 w\right|^{2\beta}$ is under special control, i.e.

$$\int_{\mathcal{R}}\left|\nabla_k\zeta\right|^2\left|\partial^2 w\right|^{2\beta}\leq\mathcal{C}\left(\left\|\varkappa w\right\|_{L^\infty(\tilde{\mathcal{G}})},A\right). \tag{4.20}$$

On the other hand, by (4.4) and Lemma 4.2 and Poincaré inequality,

$$\begin{aligned} \int_{\mathcal{R}}\xi^2\left|\nabla w\right|^{6\beta} &\leq C\int_{\mathcal{R}}\left|\nabla_k\xi\right|^2\left|\nabla w\right|^{6\beta}+C\int_{\mathcal{R}}\xi^2\left|\nabla_k\left(\nabla w\right)^{3\beta}\right|^2 \\ &\leq \mathcal{C}\left(\left\|\varkappa w\right\|_{L^\infty(\tilde{\mathcal{G}})},A\right). \end{aligned} \tag{4.21}$$

Replacing this and (4.20) into the right of (4.19) yields

$$\left|\int_{\mathcal{R}}\left(\zeta\mathcal{L}\partial^2 w\right)\left(\zeta\left(\partial^2 w\right)^{2\beta-1}\right)\right|\leq\left(\frac{CB^2}{\alpha}R_1^2+\alpha\right)\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)^{\beta}\right|^2+\mathcal{C}\left(\left\|\varkappa w\right\|_{L^\infty(\tilde{\mathcal{G}})},A\right), \tag{4.22}$$

Hence, from (4.13) we obtain

$$\begin{aligned} &\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)^{\beta}\right|^2 \\ &\leq \mathcal{C}\left(A,B,\alpha\right)\left\|\varkappa\nabla\mathbf{A}\right\|_{L^q}^2\left(\int_{\mathcal{R}}\left|\varkappa\left(\partial^2 w\right)^{\beta}\right|^p\right)^{\frac{2}{p}}+\mathcal{C}\left(\left\|\varkappa w\right\|_{L^\infty(\tilde{\mathcal{G}})},A\right), \end{aligned}$$

after we absorbed the proper term into the left. Now, as in (4.21) we have

$$\left\|\varkappa\nabla\mathbf{A}\right\|_{L^q}^q\leq C\int\varkappa^q\left|\nabla w\right|^q+C\leq\mathcal{C}\left(\left\|\varkappa w\right\|_{L^\infty(\tilde{\mathcal{G}})},A\right).$$

Replacing in the inequality above yields

$$\begin{aligned} &\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)^{\beta}\right|^2 \\ &\leq \left(\left\|\varkappa w\right\|_{L^\infty(\tilde{\mathcal{G}})},\mathbf{A}\right)\left(\int_{\mathcal{R}}\left|\varkappa\left(\partial^2 w\right)^{\beta}\right|^p\right)^{\frac{2}{p}}+\mathcal{C}\left(\left\|\varkappa w\right\|_{L^\infty(\tilde{\mathcal{G}})},A\right), \end{aligned} \tag{4.23}$$

Now, from Lemma 2.1 with $u=\partial^2 w$ and $\beta=1$, and from (4.22) with $\beta=1$, we have

$$\begin{aligned}\int_{\mathcal{R}}\left|\zeta\nabla_k\partial^2 w\right|^2 &\le C\left|\int_{\mathcal{R}}\left(\zeta\mathcal{L}\partial^2 w\right)\left(\zeta\partial^2 w\right)\right|+C\int_{\mathcal{R}}\left|\nabla_k\zeta\right|^2\left|\partial^2 w\right|^2\\ &\le \left(\frac{CB^2}{\alpha}R_1^2+\alpha\right)\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)\right|^2+\mathcal{C}\left(\|\varkappa w\|_{L^\infty(\tilde{\mathcal{G}})},A\right)+C\int_{\mathcal{R}}\left|\nabla_k\zeta\right|^2\left|\partial^2 w\right|^2\end{aligned}$$

By (4.4) the last term on the right is u.s.c., thus, after absorbing into the left the first term on the right by taking α, R_1 small, we get

$$\int_{\mathcal{R}}\left|\zeta\nabla_k\partial^2 w\right|^2\le\mathcal{C}\left(\|\varkappa w\|_{L^\infty(\tilde{\mathcal{G}})},A\right). \tag{4.24}$$

Now we take $\beta_1=\frac{2}{p}$ in (4.23) and use (4.24) to estimate the right side to obtain

$$\begin{aligned}&\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)^{\frac{2}{p}}\right|^2\\ &\le \mathcal{C}\left(\|\varkappa w\|_{L^\infty(\tilde{\mathcal{G}})},\mathbf{A}\right)\left(\int_{\mathcal{R}}\varkappa^p\left|\partial^2 w\right|^2\right)^{\frac{2}{p}}+\mathcal{C}\left(\|\varkappa w\|_{L^\infty(\tilde{\mathcal{G}})},A\right)\\ &\le \mathcal{C}\left(\|\varkappa w\|_{L^\infty(\tilde{\mathcal{G}})},\mathbf{A}\right)^{1+\frac{2}{p}}.\end{aligned}$$

We repeat this procedure by taking $\beta_N=\left(\frac{2}{p}\right)^N$,after N iterations we have

$$\begin{aligned}&\int_{\mathcal{R}}\zeta^2\left|\nabla_k\left(\partial^2 w\right)^{\left(\frac{2}{p}\right)^N}\right|^2\\ &\le \mathcal{C}\left(\|\varkappa w\|_{L^\infty(\tilde{\mathcal{G}})},\mathbf{A}\right)\left(\int_{\mathcal{R}}\varkappa^p\left|\partial^2 w\right|^{2\left(\frac{2}{p}\right)^{N-1}}\right)^{\frac{2}{p}}+\mathcal{C}\left(\|\varkappa w\|_{L^\infty(\tilde{\mathcal{G}})},A\right)\\ &\le \mathcal{C}\left(\|\varkappa w\|_{L^\infty(\tilde{\mathcal{G}})},\mathbf{A},N\right)^{1+\frac{2}{p}+\left(\frac{2}{p}\right)^2+\cdots+\left(\frac{2}{p}\right)^N}.\end{aligned}$$

This establishes (4.12) by taking N so that $\left(\frac{2}{p}\right)^N>n$, and the proof of Theorem 4.1 is complete.

4.2. **Proof of the Hypoellipticity Theorem.** In this section we provide the proof for Theorem (1.8). Given a continuous function w on $\overline{\Omega}$, $w\in H^1_{\mathrm{loc}}(\Omega,k)$, i.e.,

$$\int\varphi\left|\nabla_k w\right|^2dx<\infty,\qquad\text{for all }\varphi\in C_c^\infty(\Omega),$$

where $\nabla_k w$ is as in Definition 1.3, such that w satisfies

$$\int(\nabla\varphi)^t\,\mathbf{A}(x,w)(\nabla w)=-\int\varphi\,\vec{g}(x,w)\cdot\nabla w-\int\varphi\,f(x,w),\qquad\text{for all }\varphi\in C_c^\infty(\Omega),$$

that is, w is a weak solution of (1.13) in Ω (see Definition (1.2)), we want to prove that when $\mathbf{A}$ satisfies (1.18), $\vec{g}$ satisfies (1.20), $f_z\ge 0$ and Hipotheses (1.1) and (1.7) hold, then $w\in C^\infty(\Omega)$.

It is enough to show that for any positive integer N, there exists $R_1>0$ and $C_N>0$ such that

$$\|D^\alpha w\|_{L^\infty(\mathcal{R})}\le C_N\|w\|_{L^\infty(\Omega)},\qquad\alpha=(\alpha_1,\ldots,\alpha_n),\quad|\alpha|\le N,$$

where $\mathcal{R}$ is any cubical region in Ω, of sidelength $2R_1$. Without loss of generality we can assume $\mathcal{R}$ to be centered at the origin, that is,

$$\mathcal{R} = [-R_1, R_1]^n ,$$

we also assume $\mathcal{B} = \{x, |x| \le 2\sqrt{n}R_1\} \subset \Omega$.

For $\varepsilon > 0$, and $\mathbf{I}$ the $n \times n$ identity matrix, we define

$$\mathbf{A}_\varepsilon = \mathbf{A} + \varepsilon\, \mathbf{I}.$$

Then the operator $\mathcal{M}_\varepsilon = \operatorname{div} \mathbf{A}_\varepsilon \nabla - \vec{g} \cdot \nabla - f$ is quasilinear elliptic and satisfies the hypothesis of Theorem 15.9 in [3], so if w_ε is the solution of the Dirichlet problem

$$\mathcal{M}_\varepsilon w_\varepsilon = 0 \quad \text{in} \quad \mathcal{B}, \quad w_\varepsilon = w \quad \text{on} \quad \partial\mathcal{B},$$

then $w_\varepsilon \in C^{2+\gamma}(\mathcal{B}) \bigcap C\left(\overline{\mathcal{B}}\right)$. The classic Schauder regularity theory and a standard bootstrapping argument imply that $w_\varepsilon \in C^\infty(\mathcal{B}) \bigcap C\left(\overline{\mathcal{B}}\right)$. Now, by the maximum principle (?? in [3]), we have

$$\|w_\varepsilon\|_{L^\infty(\mathcal{B})} \le \|w_\varepsilon\|_{L^\infty(\partial\mathcal{B})} = \|w\|_{L^\infty(\mathcal{B})} = M,$$

And since the coefficients $\mathbf{A}_\varepsilon$ satisfy the hypotheses of Theorem 4.1 uniformly in $\varepsilon > 0$, we have that for any derivative D^α and $\varepsilon > 0$, the family of functions $\{D^\alpha w_\eta, \eta < \varepsilon/2\}$ is equicontinuous and uniformly bounded on $\overline{\mathcal{B}_\varepsilon}$. Therefore, there is a subsequence $\{w_{\varepsilon_i}\}$, with $\varepsilon_i \to 0$, which converges in $C^\infty(\mathcal{B})$ to a function $w_0 \in C^\infty(\mathcal{B})$. That is, $D^\alpha w_{\varepsilon_i}$ converges to $D^\alpha w_0$ uniformly on compact subsets of $\mathcal{B}$, for all multi-indexes α.

We will show that the subsequence $\{w_{\varepsilon_i}\}$ converges to w uniformly in $\overline{\mathcal{B}}$. Let x_0 be an arbitrary point on $\partial\mathcal{B}$, and let $h(x)$ be the barrier function for w at x_0 provided by Lemma 5.4, that is, for $\delta, \nu > 0$, depending only on M, , $\|\vec{g}\|_{L^\infty(\Gamma)}$, $\|f\|_{L^\infty(\Gamma)}$, the Lipschitz norm of $\mathbf{A}$ in Γ and the modulus of continuity of w in $\overline{B}$, and for $\mathcal{N}_\delta = \{y \in \mathcal{B} : |y - x_0| < \delta\}$, we have $h \in C^\infty(\mathcal{N}_\delta) \bigcap C(\mathcal{N}_\delta)$, and moreover

$$\begin{cases} h(x) \le w(x) - w(x_0), & \text{for all } x \in \mathcal{N}_\delta, \\ h(x) \le -\nu, & \text{for all } x \in \partial\mathcal{N}_\delta \bigcap \mathcal{B}, \\ h(x_0) = 0, & \\ \quad \operatorname{div} \mathbf{A}(x, u(x)) \nabla h - \vec{g}(x, u(x)) \cdot \nabla h - |f(x, u(x))| \ge 0, & \\ \quad \triangle h = \sum_{i=1}^n \partial_i^2 h \ge 0, & \text{in } \mathcal{N}_\delta, \end{cases}$$

whenever u is continuous in $\overline{B}$ and $(x, u(x)) \in \Gamma$ for all $x \in \Omega$. Now, given $\eta > 0$, there exists $\varepsilon_0 > 0$ such that $|w(x) - w(y)| < \eta$ if $|x - y| < \varepsilon_0$, $x, y \in \Omega$. Then, for $\varepsilon, \delta \le \varepsilon_0$ we have

$$-\eta < w_\varepsilon(x) - w(x_0) < \eta, \qquad \text{for all } x \in \partial\mathcal{N}_\delta \bigcap \partial\mathcal{B}, \tag{4.25}$$

and

$$\frac{2M}{\nu} h(x) \le w_\varepsilon(x) - w(x_0) \le -\frac{2M}{\nu} h(x), \text{for } x \in \partial\mathcal{N}_\delta \bigcap \mathcal{B}.$$

Therefore,

$$\frac{2M}{\nu} h(x) - \eta < w_\varepsilon(x) - w(x_0) < -\frac{2M}{\nu} h(x) + \eta, \qquad \text{for all } x \in \partial\mathcal{N}_\delta. \tag{4.26}$$

Note that for any $\varepsilon > 0$,

$$\mathcal{M}_\varepsilon h = \mathcal{M} h + \varepsilon \triangle h \ge 0.$$

We now set $\tilde{\mathbf{A}}_\varepsilon(x) = \mathbf{A}_\varepsilon(x, w_\varepsilon(x))$, $\tilde{g}_\varepsilon(x) = \vec{g}(x, w_\varepsilon(x))$, $\tilde{f}_\varepsilon(x) = f(x, w_\varepsilon(x))$ and

$$\tilde{\mathcal{M}}_\varepsilon v = \operatorname{div} \tilde{\mathbf{A}}_\varepsilon \nabla v - \tilde{g}(x) \cdot \nabla v - \tilde{f}(x).$$

Then $\tilde{\mathcal{M}}_\varepsilon(w_\varepsilon - w(x_0)) = \mathcal{M}_\varepsilon w_\varepsilon = 0$ and

$$\begin{aligned} \tilde{\mathcal{M}}_\varepsilon\left(\frac{2M}{\nu}h(x) - \eta\right) &= \frac{2M}{\nu}\left(\operatorname{div} \mathbf{A}_\varepsilon(x, w_\varepsilon(x)) \nabla h - \vec{g}(x, w_\varepsilon(x)) \cdot \nabla h\right) - f(x, w_\varepsilon(x)) \\ &\geq \frac{2M}{\nu}|f(x, w_\varepsilon(x))| - f(x, w_\varepsilon(x)) \geq 0, \end{aligned}$$

where we assumed, without loss of generality, that $\frac{2M}{\nu} \geq 1$. Therefore, we have

$$\begin{aligned} \tilde{\mathcal{M}}_\varepsilon\left(\frac{2M}{\nu}h(x) - \eta\right) &\geq \tilde{\mathcal{M}}_\varepsilon(w_\varepsilon - w(x_0)), && \text{in } \mathcal{N}_\delta \\ \frac{2M}{\nu}h(x) - \eta &\leq w_\varepsilon - w(x_0), && x \in \partial\mathcal{N}_\delta. \end{aligned}$$

From the comparison principle, Lemma 5.5, applied to the operator $\tilde{\mathcal{M}}_\varepsilon$ in $\mathcal{N}_\delta$, and from inequality (4.25), we obtain

$$\frac{2M}{\nu}h(x) - \eta \leq w_\varepsilon(x) - w(x_0), \qquad \text{for all } x \in \mathcal{N}_\delta, \quad \delta, \varepsilon \leq \varepsilon_0 = \varepsilon_0(w, \eta).$$

Since h is continuous and $h(x_0) = 0$, we conclude that for $\varepsilon_1 \leq \varepsilon_0$ small enough,

$$-2\eta < w_\varepsilon(x) - w(x_0), \qquad \text{for all } x \in \mathcal{N}_\delta, \quad \varepsilon \leq \varepsilon_1,\ \delta \leq \varepsilon_0 = \varepsilon_0(w, \eta).$$

Similarly, since $\tilde{\mathcal{M}}_\varepsilon\left(-\frac{2M}{\nu}h(x) + \eta\right) \leq 0$ we obtain

$$|w_\varepsilon(x) - w(x_0)| < 2\eta, \qquad \text{for all } x \in \mathcal{N}_\delta, \quad \varepsilon \leq \varepsilon_1,\ \delta \leq \varepsilon_0 = \varepsilon_0(w, \eta).$$

This proves our claim, and therefore, we have that $w = w_0 = \lim_{j\to\infty} w_{\varepsilon_j}$ uniformly on $\partial\mathcal{B}$. From the comparison Principle, Lemma 5.5, applied to $\mathcal{M}$ in $\mathcal{B}$, we conclude that $w = w_0$ in $\overline{\mathcal{B}}$ and therefore, w is smooth in $\mathcal{B}$, with the expected control.

5. Appendix

5.1. **Some calculations.** Let us compute the equation satisfied by $\partial^m w$. By simply commuting ∂ and $\mathcal{L}$, we have

$$\mathcal{L}(\partial^m w) = \sum_{\ell=0}^{m-1} \partial^\ell [\mathcal{L}, \partial] \partial^{m-1-\ell} w + \partial^m \mathcal{L} w. \tag{5.1}$$

We write

$$\begin{aligned} &\sum_{\ell=0}^{m-1} \partial^\ell [\mathcal{L}, \partial] \partial^{m-1-\ell} w \\ = \;& \partial^{m-1}[\mathcal{L}, \partial] w + [\mathcal{L}, \partial] \partial^{m-1} w + \delta_{m\geq 3} \sum_{\ell=1}^{m-2} \partial^\ell [\mathcal{L}, \partial] \partial^{m-1-\ell} w \\ = \;& I + II + III. \end{aligned}$$

where $\delta_{m\geq\mu} = 1$ if $m \geq \mu$ and $\delta_{m\geq\mu} = 0$ otherwise. Using the identity

$$[\mathcal{L}, \partial] = -\nabla^t (\partial \mathbf{A}) \nabla, \tag{5.2}$$

we have

$$\begin{aligned} -I &= \nabla^t \partial^{m-1}\left(\partial \mathbf{A}\right)\nabla w \\ &= \nabla^t \sum_{\ell=0}^{m-1}\tbinom{m-1}{\ell}\left(\partial^{\ell+1}\mathbf{A}\right)\nabla\left(\partial^{m-1-\ell}w\right) \\ &= \nabla^t\left(\partial^m \mathbf{A}\right)\nabla w+\nabla^t\left(\partial\mathbf{A}\right)\nabla\left(\partial^{m-1}w\right)+\delta_{m\geq 3}\left(m-1\right)\nabla^t\left(\partial^{m-1}\mathbf{A}\right)\nabla\left(\partial w\right) \\ &\quad +\delta_{m\geq 4}\left(m-1\right)\nabla^t\left(\partial^2\mathbf{A}\right)\nabla\left(\partial^{m-2}w\right)+\delta_{m\geq 5}\mathcal{P}^{\mathbf{A}}_{m+2}\left(1,\partial w,\dots,\partial^{m-1}w\right), \end{aligned}\tag{5.3}$$

where $\mathcal{P}^{\mathbf{A}}_{\ell}$ denotes a generic polynomial with coefficients depending on derivatives of $\mathbf{A}$, as given in Definition 3.1. Also from (5.2), we obtain

$$-II=\nabla^t\left(\partial\mathbf{A}\right)\nabla\partial^{m-1}w.\tag{5.4}$$

Now, using the identity

$$\begin{aligned} \partial^\ell\left[\mathcal{L},\partial\right]\partial^{m-1-\ell}w &= \left[\partial^\ell,\left[\mathcal{L},\partial\right]\right]\partial^{m-1-\ell}w+\left[\mathcal{L},\partial\right]\partial^{m-1}w \\ &= -\nabla^t\sum_{r=1}^{\ell}\tbinom{\ell}{r}\left(\partial^{r+1}\mathbf{A}\right)\nabla\left(\partial^{m-1-r}w\right)+II, \end{aligned}$$

we have

$$III=-\delta_{m\geq 3}\nabla^t\sum_{\ell=1}^{m-2}\sum_{r=1}^{\ell}\tbinom{\ell}{r}\left(\partial^{r+1}\mathbf{A}\right)\nabla\left(\partial^{m-1-r}w\right)+\delta_{m\geq 3}c\left(m\right)\,\nabla^t\left(\partial\mathbf{A}\right)\nabla\left(\partial^{m-1}w\right)$$

Adding the first and last terms separately, the sum on the left can be written as

$$\begin{aligned} &\sum_{\ell=1}^{m-2}\sum_{r=1}^{\ell}\tbinom{\ell}{r}\left(\partial^{r+1}\mathbf{A}\right)\nabla\left(\partial^{m-1-r}w\right) \\ =\;& \delta_{m\geq 4}\,c\,\left(\partial^2\mathbf{A}\right)\nabla\left(\partial^{m-2}w\right)+\sum_{r=1}^{m-2}\tbinom{m-2}{r}\left(\partial^{r+1}\mathbf{A}\right)\nabla\left(\partial^{m-1-r}w\right) \\ &+\sum_{\ell=2}^{m-3}\sum_{r=2}^{\ell}\tbinom{\ell}{r}\left(\partial^{r+1}\mathbf{A}\right)\nabla\left(\partial^{m-1-r}w\right) \\ =\;& \delta_{m\geq 4}c\left(\partial^2\mathbf{A}\right)\nabla\left(\partial^{m-2}w\right)+\left(\partial^{m-1}\mathbf{A}\right)\nabla\left(\partial w\right) \\ &+\delta_{m\geq 5}\mathcal{P}^{\mathbf{A}}_{m+1}\left(1,\partial w,\dots,\partial^{m-2}w\right), \end{aligned}$$

where $c=c\left(m\right)$. Hence, for certain constants c_i depending only on m,

$$\begin{aligned} &III \\ =\;& \delta_{m\geq 3}\left[c_1\nabla^t\left(\partial^2\mathbf{A}\right)\nabla\left(\partial^{m-2}w\right)+c_2\nabla^t\left(\partial^{m-1}\mathbf{A}\right)\nabla\left(\partial w\right)+c_3\nabla^t\left(\partial\mathbf{A}\right)\nabla\left(\partial^{m-1}w\right)\right] \\ &+\delta_{m\geq 5}\mathcal{P}^{\mathbf{A}}_{m+1}\left(1,\partial w,\dots,\partial^{m-1}w\right). \end{aligned}\tag{5.5}$$

Assembling the estimates (5.3), (5.4) and (5.5)

$$\begin{aligned}
&\sum_{\ell=0}^{m-1} \partial^{\ell}\left[\mathcal{L},\partial\right]\partial^{m-1-\ell}w \\
= \; & -\nabla^{t}\left(\partial^{m}\mathbf{A}\right)\nabla w + c_1\nabla^{t}\left(\partial\mathbf{A}\right)\nabla\left(\partial^{m-1}w\right) \\
& +c_2\delta_{m\geq 3}\nabla^{t}\left(\partial^{m-1}\mathbf{A}\right)\nabla\left(\partial w\right) + c_3\,\delta_{m\geq 4}\nabla^{t}\left(\partial^{2}\mathbf{A}\right)\nabla\left(\partial^{m-2}w\right) \\
& +\delta_{m\geq 5}\,\mathcal{P}^{\mathbf{A}}_{m+2}\left(1,\partial w,\ldots,\partial^{m-1}w\right).
\end{aligned}\tag{5.6}$$

On the other hand, it similarly follows that

$$\begin{aligned}
\partial^{m}\mathcal{L}w &= \partial^{m}\vec{g}\nabla w + \partial^{m}f \\
&= \left(\partial^{m}f\right) + \left(\partial^{m}\vec{g}\right)\cdot\nabla w + \vec{g}\cdot\nabla\left(\partial^{m}w\right) \\
&\quad +m\left(\partial\vec{g}\right)\nabla\left(\partial^{m-1}w\right) + \delta_{m\geq 3}\,c_4\left(\partial^{2}\vec{g}\right)\nabla\left(\partial^{m-2}w\right) \\
&\quad +\delta_{m\geq 4}\,m\left(\partial^{m-1}\vec{g}\right)\nabla\left(\partial w\right) + \delta_{m\geq 5}\,\mathcal{P}^{\vec{g},f}_{m+1}\left(1,\partial w,\ldots,\partial^{m-2}w\right)
\end{aligned}\tag{5.7}$$

Finally, from (5.1), (5.6) and (5.7) we obtain

$$\begin{aligned}
&\mathcal{L}\left(\partial^{m}w\right) \\
= \; & -\nabla^{t}\left(\partial^{m}\mathbf{A}\right)\nabla w + c_1\,\nabla^{t}\left(\partial\mathbf{A}\right)\nabla\left(\partial^{m-1}w\right) + f_z\left(\partial^{m}w\right) \\
& +\left(\partial^{m}w\right)\vec{g}_z\cdot\nabla w + \vec{g}\cdot\nabla\left(\partial^{m}w\right) + c_2\left(\partial\vec{g}\right)\nabla\left(\partial^{m-1}w\right) \\
& +\delta_{m\geq 3}\left[c_3\,\nabla^{t}\left(\partial^{m-1}\mathbf{A}\right)\nabla\left(\partial w\right) + c_5\,\left(\partial^{2}\vec{g}\right)\nabla\left(\partial^{m-2}w\right)\right] \\
& +\delta_{m\geq 4}\left[c_4\,\nabla^{t}\left(\partial^{2}\mathbf{A}\right)\nabla\left(\partial^{m-2}w\right) + c_6\,\left(\partial^{m-1}\vec{g}\right)\nabla\left(\partial w\right)\right] \\
& +\delta_{m\geq 5}\,\mathcal{P}^{\mathbf{A},\vec{g},f}_{m+2}\left(1,\partial w,\ldots,\partial^{m-1}w\right).
\end{aligned}\tag{5.8}$$

for all integers $m\geq 1$. For our reference, we write the special cases $m=2$ and 3:

$$\begin{aligned}
&\mathcal{L}\left(\partial^{2}w\right) \\
= \; & -\nabla^{t}\left(\partial^{2}\mathbf{A}\right)\nabla w + c_1\,\nabla^{t}\left(\partial\mathbf{A}\right)\nabla\left(\partial w\right) + f_z\left(\partial^{2}w\right) \\
& +\left(\partial^{2}w\right)\vec{g}_z\cdot\nabla w + \vec{g}\cdot\nabla\left(\partial^{2}w\right) + c_2\left(\partial\vec{g}\right)\nabla\left(\partial w\right).
\end{aligned}\tag{5.9}$$

$$\begin{aligned}
&\mathcal{L}\left(\partial^{3}w\right) \\
= \; & -\nabla^{t}\left(\partial^{3}\mathbf{A}\right)\nabla w + c_1\,\nabla^{t}\left(\partial\mathbf{A}\right)\nabla\left(\partial^{2}w\right) + f_z\left(\partial^{3}w\right) \\
& +\left(\partial^{3}w\right)\vec{g}_z\cdot\nabla w + \vec{g}\cdot\nabla\left(\partial^{3}w\right) + c_2\left(\partial\vec{g}\right)\nabla\left(\partial^{2}w\right) \\
& +c_3\,\nabla^{t}\left(\partial^{2}\mathbf{A}\right)\nabla\left(\partial w\right) + c_5\,\left(\partial^{2}\vec{g}\right)\nabla\left(\partial w\right).
\end{aligned}\tag{5.10}$$

5.2. **Quasi-linear coefficients.** Here we include a characterization of the functions $k(x,z)$ which satisfy our extra hypothesis, (1.19) and (1.23), see section 6.4 in the appendix of [8] for details.

Lemma 5.1. *Let $k(x,z)$ be a smooth nonnegative function in a bounded region $\Gamma\subset\mathbb{R}^{n}\times\mathbb{R}$, assume that for some $\gamma\geq 1$ there exists a constant $B\geq 1$ such that*

$$\left|\partial_z k(x,z)\right|\leq B\,\left(k\left(x,z\right)\right)^{\gamma},\tag{5.11}$$

then, for every $(x_0,z_0)\in\Gamma$, there exists a smooth function $f(x)$ and a Lipschitz function $h(x,z)$, with Lipschitz constant depending only on B, $\left\|k\right\|_{\infty}$ and Γ, such that

$$k\left(x,z\right)=f\left(x\right)\Big(1+f\left(x\right)^{\gamma-1}h\left(x,z\right)\Big),\tag{5.12}$$

for all (x,z) in a neighborhood of (x_0,z_0). Moreover, $h(x_0,z_0)=0$. In particular, we have

$$C^{-1}k\left(x,z'\right)\leq k(x,z)\leq Ck\left(x,z'\right),\qquad (x,z),\,\left(x,z'\right)\in\Gamma,$$

where $C=C(B,\Gamma)$. Reciprocally, if $h(x,z)$ is smooth and $f(x)$ is a smooth function such that $f(x)^{\gamma}$ is smooth, then $k(x,z)$ given by (5.12) is a smooth function which satisfies (5.11), for some $B=B(h,f,\Gamma)$.

In the next four subsections we develop the tools for proving Theorem 1.8. In 5.3 we obtain weighted L^{∞} estimates for the first derivatives of solutions of (1.6). This is done as a simple application of a priori estimates for elliptic operators, using the maximum principle applied to the equation satisfied by the derivatives (see [3], Theorem 8.17). In 4.1 we prove an a priori estimate for smooth solutions, here is where the differential algebra developed in sections 2 and 3 as a consequence of the structure of our equations comes into play. In 5.5 we establish a comparison principle adapted to our solutions. In 5.4 we construct boundary barrier functions from our particular solutions. This barriers are interesting in themselves since they require very little a priori regularity of the solution they "enclose". Finally, all this machinery then is applied in a standard approximation scheme to obtain the main result of this section.

5.3. **A priori estimates for the gradient.** In this subsection we establish a priori estimates for the gradient of solutions of (1.13). These estimates will be obtained as a consequence of the following classical result ([3] (Theorem 15.8)).

Theorem 5.2. *Let $\mathbf{A}=\left(A^{i}(y,z,q)\right)_{i=1,\dots,n}$, and $g(y,z,p)$ be smooth functions on $\Omega\times\mathbb{R}\times\mathbb{R}^{n}$ satisfying*

$$\sum_{i,j=1}^{n}\partial_{q_j}A^{i}(y,z,q)\,\xi_i\xi_j\geq\nu(|z|)\,(1+|q|)^{\tau}\,|\xi|^{2},\tag{5.13}$$

$$|\partial_q\mathbf{A}(y,z,q)|\leq\mu(|z|)\,(1+|q|)^{\tau},\tag{5.14}$$

$$\begin{aligned}(1+|q|)\,|\partial_z\mathbf{A}|+|\partial_x\mathbf{A}|+|g(y,z,q)|&\\ \leq\mu(|z|)\,(1+|q|)^{\tau+2}&,\end{aligned}\tag{5.15}$$

for all $\xi\in\mathbb{R}^{n}$, $(y,z,p)\in\Omega\times\mathbb{R}\times\mathbb{R}^{n}$ where $\tau>-1$ and ν is a positive, non-increasing function on $\mathbb{R}$ and μ is a positive, non-decreasing function on $\mathbb{R}$. Then for any $v\in C^{2}(\Omega)$ satisfying

$$\mathcal{M}v=\operatorname{div}\mathbf{A}(y,v,\nabla v)=g(y,v,\nabla v),\tag{5.16}$$

we have the estimate

$$|\nabla v(y)|\leq C,$$

for any $y\in\Omega$, where C depends on n, τ, $\nu(M_0)$, $\mu(M_0)$, M_0/d and $M_0=\sup_{B_{d(y)}}|v|$, $d=\operatorname{dist}(y,\partial\Omega)$.

Lemma 5.3. *Let $\mathbf{A}=(a_{ij}(x,z))_{i,j=1,\dots,n}$, be a smooth symmetric matrix satisfying (1.7) and (1.18):*

$$\sum_{i=1}^{n}k^{i}(x,z)\,\xi_i^{2}\leq\xi^{t}\mathbf{A}(x,z)\,\xi\leq\Lambda\sum_{i=1}^{n}k^{i}(x,z)\,\xi_i^{2},\quad\text{for all }\xi\in\mathbb{R}^{n},$$

for certain nonnegative functions k^{i}, $i=1,\dots,n$, and

$$|\partial\mathbf{A}(x,z)\,\xi|^{2}+|\partial_z\mathbf{A}(x,z)\,\xi|^{2}\leq B^{2}\xi^{t}\mathbf{A}\xi,\quad\text{for all }\xi\in\mathbb{R}^{n},$$

in a domain $\Gamma \subset \mathbb{R}^n \times I$, $\Omega \subset \mathbb{R}^n$ *open, and* $\{0\} \subset I \subset [-M_0, M_0] \subset \mathbb{R}$. *Assume there exist a constant* $B^* > 0$ *such that* $(\partial_z \mathbf{A}) \leq B^* \mathbf{A}$ *and* $(\partial_z \mathbf{A})^2 \leq {B^*}^2 \mathbf{A}^2$, *in the sense of quadratic forms, i.e.*

$$\xi^t \left(\partial_z \mathbf{A}\right) \xi \leq B\, \xi^t \mathbf{A}\, \xi, \tag{5.17}$$

and

$$\xi^t \left(\partial_z \mathbf{A}\right)^2 \xi \leq {B^*}^2 \xi^t \mathbf{A}^2 \xi, \tag{5.18}$$

for all $\xi \in \mathbb{R}^n$ *and* $1 \leq r \leq n$. *Also, let* $f(y, z, p)$ *be a smooth function on* $\Omega \times \mathbb{R} \times \mathbb{R}^n$ *satisfying*

$$f(x, z, p) \leq \mu(|z|) \left(1 + \left|p^k\right|^{\tau+1} |p|\right) \quad \text{for all } (x, z, p) \in \Gamma \times \mathbb{R}^n, \tag{5.19}$$

where $p^k = \sqrt{k^*(x,z)} \left(\sqrt{k^1(x,z)} p_1, \ldots, \sqrt{k^n(x,z)} p_n\right)$,

$$k^*(x,z) = \min\left\{k^i(x,z), 1 \leq i \leq n\right\},$$

and $\tau \geq 0$. *Then, every smooth solution* w *to*

$$\mathcal{L}w = \operatorname{div} \mathbf{A} \nabla w = f(x, w, \nabla w), \tag{5.20}$$

where $(x, w(x)) \in \Gamma$ *for all* $x \in \Omega$, *satisfies*

$$\sqrt{k^*(x,w)} \sqrt{k^i(x,w)}\, |\partial_i w| \leq C, \qquad i = 1, \ldots, n,$$

where $C = C\left(n, \Lambda, \tau, \left\|\nabla_x \vec{k}\right\|_\infty, \left\|\nabla_x^2 \vec{k}\right\|_\infty, B^*, \operatorname{dist}(x, \partial\Omega), M_0\right)$.

Proof. Let w be a smooth solution of (5.20), and let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in \Omega$ be such that $k^*(\bar{x}, \bar{z}) \neq 0$, for $\bar{z} = w(\bar{x})$. Note that it is enough to prove the lemma for $\bar{x}$ of this form, since the conclusion is otherwise trivial.

Since k^i is nonnegative and smooth, by (1.17) there exists $B > 0$ depending only on $\left\|\nabla_x^2 k^i\right\|_\infty$ and $R = \frac{1}{2} \operatorname{dist}(\bar{x}, \partial\Omega)$ such that

$$\left|\partial_j k^i(x,z)\right| \leq B \sqrt{k^i(x,z)}, \qquad \text{for all } x : |x - \bar{x}| < R, \quad (x,z) \in \Gamma \tag{5.21}$$

for $1 \leq i, j \leq n$. We set

$$W = \left\{x \in \Omega : |x_j - \bar{x}_j| \leq a \sqrt{k^*(\bar{x}, \bar{z})}\right\}, \tag{5.22}$$

where $a > 0$ is to be chosen later. Now, given $\varepsilon > 0$ and x such that $|x - \bar{x}| \leq \varepsilon$, set

$$x^i : k^i\left(x^i, \bar{z}\right) = \max\left\{k^i(y, \bar{z}) : y = \theta x + (1 - \theta)\bar{x}, 0 \leq \theta \leq 1\right\},$$

that is, x^i is the point in the segment between $\bar{x}$ and x where $k^i(\cdot, \bar{z})$ takes its maximum value Then, by the fundamental theorem of Calculus and (5.21):

$$\begin{aligned}
& k^i\left(x^i, \bar{z}\right) - k^i(\bar{x}, \bar{z}) \\
= & \int_0^1 \nabla k^i\left(\bar{x} + \theta\left(x^i - \bar{x}\right), \bar{z}\right) \cdot \left(x^i - \bar{x}\right) d\theta \\
\leq & CB\left|x^i - \bar{x}\right| \int_0^1 \sqrt{k^i\left(\bar{x} + \theta\left(x^i - \bar{x}\right), \bar{z}\right)} d\theta \\
\leq & CB\varepsilon \sqrt{k^i\left(x^i, \bar{z}\right)}.
\end{aligned}$$

Hence, if $\varepsilon = c\sqrt{k^*(\bar{x},\bar{z})}$ and $c = (2CB)^{-1}$, we have

$$\frac{1}{2}k^i\left(x^i,\bar{z}\right) \le k^i(\bar{x},\bar{z}) \le k^i\left(x^i,\bar{z}\right). \tag{5.23}$$

On the other hand, from (5.17) and Lemma 5.1 we have that for some $C = C(B^*,\Gamma) \ge 1$, and all $\xi \in \mathbb{R}^n$, (x,z), $(x,\tilde{z}) \in \Gamma$,

$$C^{-1}\,\xi^t\mathbf{A}(x,z)\,\xi \le \xi^t\mathbf{A}(x,\tilde{z})\,\xi \le C\,\xi^t\mathbf{A}(x,z)\,\xi. \tag{5.24}$$

In this sense, $\mathbf{A}$ is "almost" depends just on the variable x. From this inequality, (1.7) and (5.23); and taking $a \approx (2CB)^{-1}$, we have

$$c\bar{k}^i \le k^i(x,z) \le c^{-1}\bar{k}^i, \qquad x \in W,\ (x,z) \in \Gamma, \quad c = c(n,B), \tag{5.25}$$

where we denote $\bar{k}^i = k^i(\bar{x},\bar{z})$, $i = 1,\dots,n$.

We now consider the transformation $y = Tx$, given by

$$y = \frac{1}{\sqrt{\bar{k}^*}}\mathbf{K}(x-\bar{x}), \qquad \bar{k}_* = k^*(\bar{x},\bar{z}).$$

where $\mathbf{K} = \operatorname{diag}\left(\frac{1}{\sqrt{\bar{k}^1}},\dots,\frac{1}{\sqrt{\bar{k}^n}}\right)$. We note that if $\widetilde{W}$ is the image of W under this transformation, then

$$\tilde{W} = \left\{y : |y_j| \le (2CB)^{-1}\,\frac{1}{\sqrt{\bar{k}^j}}\right\},$$

thus $\widetilde{W}$ *contains a cube* of sidelength $\ell = \ell(C,B)$, centered at the origin. We now define

$$v(y) = w(x(y)),$$

with

$$x(y) = \left(\bar{k}^*\right)^{\frac{1}{2}}\mathbf{K}^{-1}y + \bar{x}.$$

And set

$$\tilde{\mathbf{A}}(y,v) = \mathbf{K}\,\mathbf{A}(x(y),v)\,\mathbf{K}.$$

Then, for $\xi = \mathbf{D}\mu \in \mathbb{R}^n$, we have

$$\mu^t\tilde{\mathbf{A}}\mu = \xi^t\mathbf{A}\xi \approx \sum_{i=1}^n k^i\xi_i^2 = \sum_{i=1}^n \frac{k^i(x,z)}{\bar{k}^i}\mu_i^2 \approx |\mu|^2, \tag{5.26}$$

because of (1.7) and (5.25). That is, $\tilde{\mathbf{A}}$ is *uniformly elliptic* in $\tilde{W} = T(W)$. Note that $\nabla v(y) = \nabla_\kappa w(x)$, where $\nabla_\kappa = \left(\bar{k}^*\right)^{\frac{1}{2}}\mathbf{K}^{-1}\nabla$. Now, with $\tilde{\mathbf{A}} = (\tilde{a}_{i,j})_{i,j=1\dots,n}$, we compute

$$\begin{aligned}
\operatorname{div}\tilde{\mathbf{A}}\nabla v &= \sum_{i,j}\partial_{y_i}\tilde{a}_{i,j}(y,v)\,\partial_{y_j}v \\
&= \left(\bar{k}^*\right)^{\frac{1}{2}}\sum_{i,j}\left(\bar{k}^j\right)^{\frac{1}{2}}\partial_{y_i}\tilde{a}_{i,j}(y,v)\,\partial_j w \\
&= \bar{k}^*\sum_{i,j}\left(\bar{k}^j\right)^{\frac{1}{2}}\left(\bar{k}^i\right)^{\frac{1}{2}}\partial_i\frac{a_{i,j}(x,w)}{\left(\bar{k}^j\right)^{\frac{1}{2}}\bar{k}_i^{\frac{1}{2}}}\partial_j w \\
&= k^*(\bar{x},\bar{z})\,\mathcal{L}w(x(y),v),
\end{aligned}$$

and thus, v is a solution of

$$\mathcal{N}v = \operatorname{div}\tilde{\mathbf{A}}\nabla v = g(y,v,\nabla v), \qquad y \in \widetilde{W}, \tag{5.27}$$

where

$$g\left(y,v,\nabla v\right)=\bar{k}^{*}\,f\left(x\left(y\right),v,\nabla^{\kappa}v\right),$$

where $\nabla^{\kappa}=\left(\bar{k}^{*}\right)^{-\frac{1}{2}}\mathbf{K}\nabla$. Note that the operator $\mathcal{N}$ is of the form (5.16) with $A^{i}\left(y,z,q\right)=\sum_{j=1}^{n}\tilde{a}_{ij}\left(y,z\right)q_{j}$. We claim that $\mathcal{N}$ satisfies the hypotheses (5.13), (5.14) and (5.15) of Theorem (5.2). Hence

$$\left\Vert \nabla v\right\Vert_{L^{\infty}\left(\frac{1}{2}\widetilde{W}\right)}\leq C\left(n,\tau,\nu\left(M_{1}\right),\mu\left(M_{1}\right),B,B^{*},M_{1}\right),$$

where $M_{1}=\sup_{y\in\tilde{W}}\left|v\right|=\sup_{x\in W}\left|w\right|\leq M_{0}$. This implies

$$\sqrt{k^{*}\left(x,w\right)}\sqrt{k^{i}\left(x,w\right)}\left|\partial_{i}w\left(x\right)\right|\leq C,\qquad x\in\tfrac{1}{2}W,\quad i=1,\ldots,n,$$

as wanted.

So it only rests to establish our claim. Since

$$\sum_{i,j=1}^{n}\partial_{q_{j}}A^{i}\left(y,z,q\right)\xi_{i}\xi_{j}=\sum_{i,j=1}^{n}\tilde{a}_{ij}\left(y,z\right)\xi_{i}\xi_{j}\approx\left|\xi\right|^{2},$$

because of (5.26), we have that $\mathcal{N}$ satisfies (5.13). Also, since $\partial_{q_{j}}A_{i}\left(y,z,q\right)=\tilde{a}_{i,j}\left(y,z\right)$, it is obvious that $\mathcal{N}$ satisfies (5.14). Now, from (5.19) we have

$$\begin{aligned}
& \left|g\left(y,z,q\right)\right|\\
= & \; k^{*}\left(\bar{x},\bar{z}\right)f\left(x\left(y\right),v,\left(\bar{k}^{*}\right)^{-\frac{1}{2}}\mathbf{K}\,q\right)\\
\leq & \; \mu\left(\left|z\right|\right)\left(\bar{k}^{*}+\left|\sqrt{\frac{k^{*}\left(x,z\right)}{\bar{k}^{*}}}\left(\sqrt{\frac{k^{1}\left(x,z\right)}{\bar{k}^{1}}}q_{1},\ldots,\sqrt{\frac{k^{n}\left(x,z\right)}{\bar{k}^{n}}}q_{n}\right)\right|^{\tau+1}\left|q\right|\right).
\end{aligned}$$

And, from (5.25) we obtain

$$\left|g\left(y,z,q\right)\right|\leq C\,c^{-\left(\tau+1\right)}\mu\left(\left|z\right|\right)\left(1+\left|q\right|^{\tau+2}\right). \tag{5.28}$$

On the other hand, from (5.18) and (5.26),

$$\begin{aligned}
\sum_{i=1}^{n}\left|\partial_{z}A_{i}\right|^{2} = & \sum_{i=1}^{n}\left|\sum_{j=1}^{n}\partial_{z}\tilde{a}_{ij}\left(y,z\right)q_{j}\right|^{2}=\sum_{j,l=1}^{n}q_{l}\left(\sum_{i=1}^{n}\partial_{z}\tilde{a}_{li}\left(y,z\right)\partial_{z}\tilde{a}_{ij}\left(y,z\right)\right)q_{j}\\
= & \sum_{l=1}^{n}\sum_{j=1}^{n}\frac{q_{l}}{\left(\bar{k}^{l}\right)^{\frac{1}{2}}}\left(\sum_{i=1}^{n}\frac{\partial_{z}a_{li}\left(x\left(y\right),z\right)\partial_{z}a_{ij}\left(x\left(y\right),z\right)}{\sqrt{\bar{k}^{i}}\sqrt{\bar{k}^{i}}}\right)\frac{q_{j}}{\left(\bar{k}^{j}\right)^{\frac{1}{2}}}\\
= & \; q^{t}\left(\mathbf{K}\left(\partial_{z}\mathbf{A}\right)\mathbf{K}\right)^{2}q\\
\leq & \; B^{*^{2}}q^{t}\left(\mathbf{KAK}\right)^{2}q=B^{*^{2}}q^{t}\left(\tilde{\mathbf{A}}\right)^{2}q\leq CB^{*^{2}}\left|q\right|^{2}. \qquad (5.29)
\end{aligned}$$

Finally, from (1.18) and (5.26),

$$\begin{aligned}
\sum_{i=1}^{n} \left|\nabla_y A_i\right|^2 &= \sum_{i=1}^{n}\sum_{\ell=1}^{n}\left(\frac{\partial}{\partial y_\ell}\sum_{j=1}^{n}\tilde{a}_{ij}\left(y,z\right)q_j\right)^2 \\
&= \sum_{i=1}^{n}\sum_{\ell=1}^{n}\frac{\bar{k}^*\bar{k}^\ell}{\bar{k}^i}\left(\sum_{j=1}^{n}\frac{\partial a_{ij}}{\partial x_\ell}\frac{q_j}{\left(\bar{k}^j\right)^{\frac{1}{2}}}\right)^2 \\
&\leq \sum_{\ell=1}^{n}\bar{k}^\ell\left|\left(\partial_\ell \mathbf{A}\right)\mathbf{K}q\right|^2 \\
&\leq B^2\sum_{\ell=1}^{n}\bar{k}^\ell q^t\mathbf{K}\mathbf{A}\,\mathbf{K}q \\
&= B^2\sum_{\ell=1}^{n}\bar{k}^\ell\, q^t\tilde{\mathbf{A}}\,q \\
&\leq CB^2\left|q\right|^2.
\end{aligned}$$

From this inequality, (5.28) and (5.29) it follows that $\mathcal{N}$ satisfies (5.15) with $\tau = 0$, which completes the proof of the lemma.

5.4. **Barriers for the Dirichlet problem.** On this section we present barriers for continuous weak solutions of the Dirichlet problem in a smooth convex domain. The interesting aspect of the barriers constructed below is that even though they are specialized to a particular solution w, they only require continuity of w, while the assumption $w \in H^1\left(\Omega, k\right)$ is just used to make sense of the equation $\mathcal{L}w = 0$.

Lemma 5.4. *Let Ω be a convex domain with boundary $\partial\Omega$ with positive Gaussian curvature bounded below by $\gamma_0 > 0$. Let $w \in H^1\left(\Omega, k\right)$ be a weak solution of*

$$\mathcal{L}w = \vec{g}\left(x,w\right)\cdot\nabla w + f\left(x,w\right) \qquad \text{in } \Omega.$$

for $\mathcal{L}$ as in (1.13). Let Γ satisfy $\Omega\times\left[-M,M\right]\subset\Gamma\subset\Omega\times\mathbb{R}$. If $\mathbf{A}\left(x,z\right) = \left(a^{ij}\left(x,z\right)\right)_{i,j=1}^{n}$ is Lipschitz in Γ, $\vec{g}\left(x,z\right) = \left(g^1\left(x,z\right),\ldots,g^n\left(x,z\right)\right)$, $f\left(x,z\right)$ are bounded in Γ, and there exists a positive μ such that

$$\xi^t\mathbf{A}\left(x,z\right)\xi \geq \mu\left|\xi^2\right|, \qquad \text{for all } x\in\partial\Omega, \quad \left(x,z\right)\in\bar{\Gamma}. \tag{5.30}$$

If also w is continuous in $\overline{\Omega}$, with $\|w\|_{L^\infty(\Omega)} \leq M$, then for every boundary point P, there exists a function $h \in C\left(\overline{\Omega}\right)\bigcap C^\infty\left(\Omega\right)$, a positive constant $\nu = \nu\left(w\right)$ and a neighborhood $\mathcal{N}$ of P, such that for any $u \in H^1\left(\Omega,k\right)\bigcap C\left(\overline{\Omega}\right)$ satisfying $\left(x,u\left(x\right)\right)\in\Gamma$ for all $x\in\Omega$,

$$\begin{cases}
h\left(Q\right) \leq w\left(Q\right) - w\left(P\right), & \text{for all } Q\in\Omega\bigcap\mathcal{N}, \\
h\left(Q\right) \leq -\nu, & \text{for all } Q\in\Omega\bigcap\partial\mathcal{N}, \\
h\left(P\right) = 0, & \\
\quad \operatorname{div}\mathbf{A}\left(x,u\left(x\right)\right)\nabla h - \vec{g}\left(x,u\left(x\right)\right)\cdot\nabla h - \left|f\left(x,u\left(x\right)\right)\right| \geq 0, & \\
\quad \triangle h = \sum_{i=1}^{n}\partial_i^2 h \geq 0, & \text{in } \Omega\bigcap\mathcal{N}.
\end{cases}$$

The radius of $\mathcal{N}$ depends only on M, $\|\vec{g}\|_{L^\infty(\Gamma)}$, $\|f\|_{L^\infty(\Gamma)}$, the Lipschitz norm of $\mathbf{A}$ in Γ and the modulus of continuity of w in $\overline{\Omega}$.

Proof. Let $P \in \partial\Omega$, translating the solution and the coefficients we may assume without loss of generality P to be the origin. In that case $\partial\Omega$ is given in local coordinates by

$$y_n = \gamma \left|y'\right|^2 + O\left(\left|y'\right|^3\right), \tag{5.31}$$

with $\gamma \geq \gamma_0 > 0$, and $y_i = \sum_{j=1}^n p_{ij} x_j$, where

$$P = \left(p_{ij}\right)_{i,j=1}^n, \text{ is a rotation matrix.}$$

Since $\partial_i = \partial_{x_i} = \sum_{j=1}^n p_{ij}\, \partial_{y_j}$, we have that $\operatorname{div} \mathbf{A}(x,u) \nabla v$ is given is given in terms of the new coordinates as

$$\begin{aligned}
\operatorname{div} \mathbf{A}(x,u)\nabla v &= \sum_{i,j=1}^n \partial_i \left(a^{ij}(x,u)\,\partial_j v\right) = \sum_{i,j,\ell=1}^n \partial_i\left(a^{ij} p_{\ell j}\left(\partial_{y_\ell} v\right)\right) \\
&= \sum_{i,j,\ell=1}^n \left(\partial_i a^{ij}\right) p_{\ell j}\left(\partial_{y_\ell} v\right) + \sum_{i,j,\ell=1}^n a^{ij} p_{\ell j}\left(\partial_i\left(\partial_{y_\ell} v\right)\right) \\
&= \sum_{i,j,\ell=1}^n \left(\left(\partial_i a^{ij}\right) + \left(\partial_z a^{ij}\right)\sum_{m=1}^n p_{mi}\left(\partial_{y_m} u\right)\right) p_{\ell j}\left(\partial_{y_\ell} v\right) + \sum_{i,j.\ell,m=1}^n p_{\ell j} a^{ij} p_{mi}\left(\partial_{y_m}\partial_{y_\ell} v\right) \\
&= \sum_{i,j,\ell=1}^n a_i^{ij} p_{\ell j}\left(\partial_{y_\ell} v\right) + \sum_{i,j,\ell,m=1}^n a_z^{ij} p_{mi} p_{\ell j}\left(\partial_{y_m} u\right)\left(\partial_{y_\ell} v\right) + \sum_{i,j,\ell,m=1}^n p_{\ell j} a^{ij} p_{mi}\left(\partial_{y_m}\partial_{y_\ell} v\right).
\end{aligned} \tag{5.32}$$

We let $\phi(r)$ be the modulus of continuity of w in $\overline{\Omega}$, that is

$$\phi(r) = \sup_{x,y\in\Omega, |x-y|\leq r} \left|w(x) - w(y)\right|.$$

Then, ϕ is continuous, $\phi(0) = 0$, and (by taking a bigger ϕ if necessary) we may assume ϕ to be concave, strictly increasing on $[0,R]$, and smooth on $(0,R]$ where $R > 0$ is the diameter of Ω, hence ϕ satisfies

$$\phi(r) \geq a\,r, \qquad r \in [0,R],\ a = \frac{\phi(R)}{R}, \tag{5.33}$$

$$\liminf_{r\to 0^+} \phi'(r) \geq a > 0, \tag{5.34}$$

$$\phi''(r) \leq 0, \qquad r \in [0,R]. \tag{5.35}$$

Let $0 < \alpha_0 \leq 1$ to be determined later, and set $r_0 = \phi^{-1}(\alpha_0)$ if $\phi(R) > \alpha_0$, and $r_0 = R$ otherwise, we now set

$$\psi(r) = \sqrt{\phi(r)}, \qquad 0 \leq r \leq r_0.$$

Then, ψ satisfies

$$\sqrt{\alpha_0} \geq \psi(r) \geq (\psi(r))^2 = \phi(r). \tag{5.36}$$

and because of (5.34) and (5.35) we have

$$\begin{aligned}
-\psi''(r) &= \frac{-2\phi''(r)\,\phi(r) + \left(\phi'(r)\right)^2}{4\left(\phi(r)\right)^{\frac{3}{2}}} \\
&\geq \frac{\left(\phi'(r)\right)^2}{4\left(\phi(r)\right)^{\frac{3}{2}}} = \frac{\left(\psi'(r)\right)^2}{\psi(r)},
\end{aligned}$$

that is,

$$\left(\psi'(r)\right)^2 \leq -\psi(r)\,\psi''(r), \qquad 0 \leq r \leq r_0. \tag{5.37}$$

Moreover, from $\phi(0) = 0$ and (5.34) it follows that

$$\lim_{r\to 0+} \psi'(r) = \infty. \tag{5.38}$$

We now define

$$h(y) = -\,\psi\left(2\sqrt{\frac{y_n}{\gamma}}\right),$$

where $N \geq 1$ is to be specified later. Now set $\mathcal{N}_\delta = \Omega \bigcap \{y_n < \delta\}$, $0 < \delta < R$, then from (5.31) it follows that for $y \in \mathcal{N}_\delta$ and δ small enough, $2\sqrt{\frac{y_n}{\gamma}} \geq |y|$, therefore, from (5.36) we get

$$\begin{aligned} h(y) - (w(y) - w(0)) &= w(0) - w(y) - \psi\left(2\sqrt{\tfrac{y_n}{\gamma}}\right) \\ &\leq \phi(|y|) - \psi\left(2\sqrt{\tfrac{y_n}{\gamma}}\right) \\ &\leq (\psi(|y|))^2 - \psi(|y|) \leq 0. \end{aligned} \tag{5.39}$$

We notice that

$$\begin{aligned} \partial_{y_n} h(y) &= -\sqrt{\frac{1}{\gamma y_n}}\psi'\left(2\sqrt{\frac{y_n}{\gamma}}\right) \\ \partial^2_{y_n} h(y) &= -\frac{1}{\gamma y_n}\psi''\left(2\sqrt{\frac{y_n}{\gamma}}\right) + \frac{1}{2}\frac{1}{\sqrt{\gamma} y_n^{\frac{3}{2}}}\psi'\left(2\sqrt{\frac{y_n}{\gamma}}\right) \\ \partial_{y_j} h(y) &= 0 \qquad j \neq n, \end{aligned}$$

in particular,

$$\triangle h = \sum_{i=1}^{n} \partial_i^2 h \geq 0. \tag{5.40}$$

Replacing w by h in (5.32), we obtain

$$\begin{aligned} \operatorname{div} \mathbf{A}(x,u)\nabla h &= -\sqrt{\frac{1}{\gamma y_n}}\psi' \sum_{i,j=1}^{n} a_i^{ij} p_{nj} - \sqrt{\frac{1}{\gamma y_n}}\psi' \sum_{i,j,m=1}^{n} a_z^{ij} p_{mi} p_{nj} (\partial_{y_m} u) \\ &\quad -\frac{1}{\gamma y_n}\psi'' \sum_{i,j} p_{nj} a^{ij} p_{ni} + \frac{1}{2}\frac{1}{\sqrt{\gamma} y_n^{\frac{3}{2}}}\psi' \sum_{i,j} p_{nj} a^{ij} p_{ni}. \end{aligned} \tag{5.41}$$

Since $\mathbf{A}$ is smooth, and $\mathbf{P}$ is a rotation matrix, we have

$$\left|\sum_{i,j} a_i^{ij}(x,u)\, p_{nj}\right| + \left|\sum_{i,j} a_z^{ij}(x,u)\, p_{ni} p_{nj}\right| \leq C.$$

Replacing in (5.41) we obtain

$$\operatorname{div} \mathbf{A}(x,u)\nabla h \geq -\frac{C}{\sqrt{\gamma y_n}}\psi' + \left(-\frac{1}{\gamma y_n}\psi'' + \frac{1}{2}\frac{1}{\sqrt{\gamma} y_n^{\frac{3}{2}}}\psi'\right) \sum_{i,j} p_{nj} a^{ij} p_{ni}.$$

From (5.30) and the continuity of $\mathbf{A}$ it follows

$$\sum_{i,j} p_{nj} a^{ij}(x,u)\, p_{ni} \geq \mu\, |p_n|^2 = \mu > 0,$$

hence,

$$\operatorname{div} \mathbf{A}(x,u)\nabla h \geq -\frac{\mu}{\gamma y_n}\psi'' + \frac{C}{\sqrt{\gamma y_n}}\psi'\left(\frac{\mu}{2C}\frac{1}{y_n} - 1\right). \tag{5.42}$$

Now, since for $x \in \Omega$ we have $(x, u(x)) \in \Gamma$, and $|\vec{g}|$ and $|f|$ are bounded in Γ, we have

$$|\vec{g}(x,u)\cdot\nabla h| \leq C\sqrt{\frac{1}{\gamma y_n}}\psi', \qquad x \in \mathcal{N}_\delta \tag{5.43}$$

$$|f(x,u)| \leq C, \qquad x \in \mathcal{N}_\delta. \tag{5.44}$$

from (5.42), (5.43), (5.44) and (5.37) we obtain

$$\operatorname{div} \mathbf{A}(x,u)\nabla h - \vec{g}(x,u)\cdot\nabla h - |f(x,u)| \geq -\frac{\mu}{\gamma y_n}\psi'' + \frac{C}{\sqrt{\gamma y_n}}\psi'\left(\frac{\mu}{2C}\frac{1}{y_n} - 1\right) - C$$

for all $x \in \mathcal{N}_\delta$. Since for δ small, we have $\frac{\mu}{2C}\frac{1}{y_n} - 1 \geq 1$, by (5.38) ψ' tends to $+\infty$ as $\delta \to 0$, we have that for δ small enough,

$$\operatorname{div} \mathbf{A}(x,u)\nabla h - \vec{g}(x,u)\cdot\nabla h - |f(x,u)| \geq 0 \qquad x \in \mathcal{N}_\delta.$$

Finally, we set $\nu = \psi\left(2\sqrt{\frac{\delta}{\gamma}}\right)$, then obviously $h(y) \leq -\nu$ on $\Omega \bigcap \partial\mathcal{N}_\delta$. The inequality above, together with (5.39) and (5.40), finish the proof of the lemma.

5.5. **A Comparison Principle.**

Lemma 5.5. *Let w_0 and w_1 be on the weighted Sobolev $H^1(\Omega, k)$, i.e., w_0 and w_1 satisfy*

$$\|\nabla_k w_0\|^2_{L^2(\Omega)} = \int_\Omega (\nabla w_0)^t \mathbf{A}\nabla w_0^2 < \infty, \qquad \|\nabla_k w_1\|^2_{L^2(\Omega)} = \int_\Omega (\nabla w_1)^t \mathbf{A}\nabla w_1^2 < \infty.$$

Suppose that w_0, $w_1 \in C(\overline{\Omega})$, $w_0 \geq w_1$ on $\partial\Omega$ and

$$\mathcal{M}(w_1) \geq \mathcal{M}(w_0) \quad \text{in } \Omega, \text{ or} \tag{5.45}$$

where

$$\mathcal{M}(w) = \operatorname{div} \mathbf{A}(x,w)\nabla w - \vec{g}(x,w)\cdot\nabla w - f(x,w).$$

We assume that $\mathbf{A}$ satisfies the following conditions

$$\xi^t(\partial_z\mathbf{A})\xi \leq B^2\, \xi^t\mathbf{A}\,\xi, \qquad \xi \in \mathbb{R}^n, \tag{5.46}$$

$$\xi^t(\partial_z\mathbf{A})^2\,\xi \leq B^2\, \xi^t\mathbf{A}^2\,\xi, \qquad \xi \in \mathbb{R}^n, \tag{5.47}$$

$\vec{g}$ satisfies

$$|\vec{g}(x,z)\cdot\xi|^2 \leq B^2\, \xi^t\mathbf{A}\xi, \qquad \xi \in \mathbb{R}^n, \tag{5.48}$$

$$|(\partial_z\vec{g})\cdot\xi|^2 \leq B^2\, \xi^t\mathbf{A}\,\xi, \qquad \xi \in \mathbb{R}^n, \tag{5.49}$$

and f is nondecreasing in z, i.e.

$$f_z(x,z) \geq 0, \qquad (x,z) \in \Gamma. \tag{5.50}$$

Then $w_0 \geq w_1$ in Ω. In particular, if $\mathcal{M}w_0 = \mathcal{M}w_1$ in Ω, and $w_0 = w_1$ on $\partial\Omega$, then $w_0 = w_1$ in $\overline{\Omega}$.

Proof. From (5.45) we have

$$\begin{aligned} 0 &\leq -\int_\Omega \langle \mathcal{M}(w_0) - \mathcal{M}(w_1), \varphi\rangle \\ &= \int_\Omega \left[\mathbf{A}(x,w_0)\nabla w_0 - \mathbf{A}(x,w_1)\nabla w_1\right] \nabla\varphi \\ &\quad + \int_\Omega \left[\vec{g}(x,w_0) w_0 - \vec{g}(x,w_1) w_1\right] \nabla\varphi \\ &\quad + \int_\Omega \left[f(x,w_0) - f(x,w_1)\right] \varphi, \end{aligned} \tag{5.51}$$

whenever $\varphi \in H^1(\Omega, k)$ is nonnegative. Now let $u = w_1 - w_0$, and set

$$\varphi(x) = \frac{u^+}{u^+ + \varepsilon},$$

so

$$\nabla\varphi(x) = \varepsilon \frac{\nabla u^+}{(u^+ + \varepsilon)^2}$$

where $u^+ = \max\{-u, 0\}$. We note that since w_0 and w_1 are continuous in $\overline{\Omega}$, then φ is continuous in Ω. We claim that $\varphi \in H^1(\Omega, k)$. Indeed, we have

$$\int_\Omega |\nabla_k \varphi|^2 = \varepsilon^2 \int_\Omega \frac{(\nabla u^+)^t \mathbf{A} \nabla u^+}{(u^+ + \varepsilon)^4} \leq \frac{C}{\varepsilon^2} \|u\|_{H^1(\Omega,k)} < \infty,$$

hence $\varphi \in H^1(\Omega, k)$. Since obviously $\varphi \geq 0$, we can replace φ into inequality (5.51) to obtain

$$\begin{aligned} 0 \leq{}& \varepsilon \int_\Omega \left[\mathbf{A}(x,w_0)\nabla w_0 - \mathbf{A}(x,w_1)\nabla w_1\right] \frac{\nabla u^+}{(u^+ + \varepsilon)^2} \\ &+ \varepsilon \int_\Omega \left[\vec{g}(x,w_0) w_0 - \vec{g}(x,w_1) w_1\right] \frac{\nabla u^+}{(u^+ + \varepsilon)^2} + \int_\Omega \left[f(x,w_0) - f(x,w_1)\right] \frac{u^+}{u^+ + \varepsilon}. \end{aligned} \tag{5.52}$$

Now, if $w_t = t w_1 + (1-t) w_0$, $0 \leq t \leq 1$ (note that w_t is well-defined at the end-points) then $(\partial_t w_t) = u$, and from the fundamental theorem of calculus we have

$$\begin{aligned} \mathbf{A}(x,w_1)\nabla w_1 - \mathbf{A}(x,w_0)\nabla w_0 &= \int_0^1 \partial_t \left\{\mathbf{A}(x,w_t)\nabla w_t\right\} dt \\ &= u \left\{\int_0^1 \mathbf{A}_z(x,w_t)\nabla w_t \, dt\right\} + \left\{\int_0^1 \mathbf{A}(x,w_t)\, dt\right\} \nabla u, \end{aligned}$$

$$\begin{aligned} \vec{g}(x,w_1) w_1 - \vec{g}(x,w_0) w_0 &= \int_0^1 \partial_t \left\{\vec{g}(x,w_t) w_t\right\} dt \\ &= u \left\{\int_0^1 \vec{g}_z(x,w_t) w_t \, dt\right\} + u \left\{\int_0^1 \vec{g}(x,w_t)\, dt\right\}. \end{aligned}$$

and, because of (5.50),

$$\left[f(x,w_0) - f(x,w_1)\right] \frac{u^+}{u^+ + \varepsilon} = -\frac{(u^+)^2}{u^+ + \varepsilon} \int_0^1 f_z(x,w_t)\, dt \leq 0$$

Setting

$$\begin{aligned}\vec{a}\left(x\right) &= \left\{\int_0^1 \mathbf{A}_z\left(x,w_t\right)\nabla w_t\ dt\right\}, \qquad \tilde{\mathbf{A}}\left(x\right)=\left\{\int_0^1 \mathbf{A}\left(x,w_t\right)\ dt\right\},\\ \bar{g}\left(x\right) &= \left\{\int_0^1 \vec{g}_z\left(x,w_t\right)w_t\ dt\right\}, \qquad \tilde{g}\left(x\right)=\left\{\int_0^1 \vec{g}\left(x,w_t\right)dt\right\}.\end{aligned}$$

Plugging the above expressions into inequality (5.52), and using that on the support of u^+ $\Omega^+=\{x\in\Omega: u\left(x\right)\geq 0\}$ we have $u=u^+$, yields

(5.53)
$$\varepsilon\int_\Omega \frac{\left(\tilde{\mathbf{A}}\left(x\right)\nabla u^+\right)\cdot\nabla u^+}{\left(u^++\varepsilon\right)^2}\leq -\varepsilon\int_\Omega \frac{u^+\,\vec{a}\left(x\right)\cdot\nabla u^+}{\left(u^++\varepsilon\right)^2}-\varepsilon\int_\Omega u^+\left(\,\bar{g}\left(x\right)+\,\tilde{g}\left(x\right)\right)\,\frac{\nabla u^+}{\left(u^++\varepsilon\right)^2}.$$

Now, from Schwartz inequality and (5.46):

$$\left|\left(\mathbf{A}_z\left(x,w_t\right)\nabla w_t\right)\cdot\nabla\left(u^+\right)\right|\leq\frac{\alpha}{u^+}\left(\mathbf{A}\left(x,w_t\right)\nabla\left(u^+\right)\right)\cdot\nabla\left(u^+\right)+\frac{CB^2}{\alpha}u^+\left(\nabla w_t\right)^t\mathbf{A}\left(x,w_t\right)\nabla w_t.$$

Hence,

$$\begin{aligned}\left|\int_\Omega \frac{u^+\,\vec{a}\left(x\right)\cdot\nabla u^+}{\left(u^++\varepsilon\right)^2}\right| &\leq \int_\Omega\left\{\int_0^1\left|\mathbf{A}_z\left(x,w_t\right)\nabla w_t\cdot\nabla u^+\right|\ dt\right\}\frac{u^+}{\left(u^++\varepsilon\right)^2}\\ &\leq \alpha\int_\Omega\left(\left\{\int_0^1\mathbf{A}\left(x,w_t\right)dt\right\}\nabla\left(u^+\right)\right)\cdot\nabla\left(u^+\right)\frac{1}{\left(u^++\varepsilon\right)^2}\\ &\quad+\frac{CB^2}{\alpha}\int_\Omega\left\{\int_0^1\left(\nabla w_t\right)^t\mathbf{A}\left(x,w_t\right)\nabla w_t\ dt\right\}\frac{\left(u^+\right)^2}{\left(u^++\varepsilon\right)^2}\\ &\leq \alpha\int_\Omega\frac{\left(\tilde{\mathbf{A}}\left(x\right)\nabla\left(u^+\right)\right)\cdot\nabla\left(u^+\right)}{\left(u^++\varepsilon\right)^2}+\frac{CB^2}{\alpha}, \qquad (5.54)\end{aligned}$$

where we used that $w_t\in H^1\left(\Omega,k\right)$. Now, from our hypothesis (5.47) on $\mathbf{A}_z$ and Lemma 5.1, we have that for some positive $C=C\left(B,\Gamma\right)$, and all $\xi\in\mathbb{R}^n$, $\left(x,z\right)$, $\left(x,\tilde{z}\right)\in\Gamma$,

$$C^{-1}\,\xi^t\mathbf{A}\left(x,z\right)\xi\leq\xi^t\mathbf{A}\left(x,\tilde{z}\right)\xi\leq C\,\xi^t\mathbf{A}\left(x,z\right)\xi. \qquad (5.55)$$

Then, from (5.48), (5.49) and the fact that w_t is bounded for all $0 \leq t \leq 1$, it follows that

$$\begin{aligned}
&\int_\Omega u^+ \left\{ \int_0^1 \vec{g}_z\left(x,w_t\right) w_t \; dt \right\} \frac{\nabla u^+}{\left(u^+ + \varepsilon\right)^2} \\
\leq \quad & \int_\Omega \left\{ \int_0^1 \left| w_t \, \vec{g}_z\left(x,w_t\right) \cdot \nabla u^+ \right| \; dt \right\} \frac{u^+}{\left(u^+ + \varepsilon\right)^2} \\
\leq \quad & \int_\Omega \left\{ \int_0^1 \left| \vec{g}_z\left(x,w_t\right) \cdot \nabla u^+ \right|^2 \; dt \right\}^{\frac{1}{2}} \left\{ \int_0^1 \left| w_t \right|^2 \; dt \right\}^{\frac{1}{2}} \frac{u^+}{\left(u^+ + \varepsilon\right)^2} \\
\leq \quad & C \int_\Omega \left\{ \int_0^1 \left(\left(\mathbf{A}\left(x,w_t\right) \nabla u^+ \right) \cdot \nabla u^+ \right) \; dt \right\}^{\frac{1}{2}} \frac{u^+}{\left(u^+ + \varepsilon\right)^2} \\
\leq \quad & C \int_\Omega \left(\left(\tilde{\mathbf{A}}\left(x\right) \nabla\left(u^+\right) \right) \cdot \nabla\left(u^+\right) \right)^{\frac{1}{2}} \frac{u^+}{\left(u^+ + \varepsilon\right)^2} \\
\leq \quad & \alpha \int_\Omega \frac{\left(\tilde{\mathbf{A}}\left(x\right) \nabla\left(u^+\right) \right) \cdot \nabla\left(u^+\right)}{\left(u^+ + \varepsilon\right)^2} + \frac{CB^2}{\alpha}, \qquad (5.56)
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega u^+ \left\{ \int_0^1 \vec{g}\left(x,w_t\right) dt \right\} \frac{\nabla u^+}{\left(u^+ + \varepsilon\right)^2} \\
\leq \quad & \int_\Omega \left\{ \int_0^1 \left| \vec{g}\left(x,w_t\right) \cdot \nabla u^+ \right| dt \right\} \frac{u^+}{\left(u^+ + \varepsilon\right)^2} \\
\leq \quad & B \int_\Omega \left\{ \int_0^1 \left(\left(\mathbf{A}\left(x,w_t\right) \nabla\left(u^+\right) \right) \cdot \nabla\left(u^+\right) \right)^{\frac{1}{2}} dt \right\} \frac{u^+}{\left(u^+ + \varepsilon\right)^2} \\
\leq \quad & B \int_\Omega \left(\left(\tilde{\mathbf{A}}\left(x\right) \nabla\left(u^+\right) \right) \cdot \nabla\left(u^+\right) \right)^{\frac{1}{2}} \frac{u^+}{\left(u^+ + \varepsilon\right)^2} \\
\leq \quad & \alpha \int_\Omega \frac{\left(\tilde{\mathbf{A}}\left(x\right) \nabla\left(u^+\right) \right) \cdot \nabla\left(u^+\right)}{\left(u^+ + \varepsilon\right)^2} + \frac{CB^2}{\alpha}. \qquad (5.57)
\end{aligned}$$

Replacing (5.54)–(5.57) into (5.53), and dividing by ε both sides, we get

$$\int_\Omega \frac{\left(\tilde{\mathbf{A}}\left(x\right) \nabla u^+ \right) \cdot \nabla u^+}{\left(u^+ + \varepsilon\right)^2} \leq \alpha \int_\Omega \frac{\left(\tilde{\mathbf{A}}\left(x\right) \nabla\left(u^+\right) \right) \cdot \nabla\left(u^+\right)}{\left(u^+ + \varepsilon\right)^2} + \frac{C}{\alpha},$$

absorbing the first term on the right into the left we obtain

$$\int_\Omega \frac{\left(\tilde{\mathbf{A}}\left(x\right) \nabla u^+ \right) \cdot \nabla u^+}{\left(u^+ + \varepsilon\right)^2} \leq C < \infty,$$

uniformly in $\varepsilon > 0$, hence, from (5.55) we conclude

$$\int_\Omega \frac{\left| \nabla_k u^+ \right|^2}{\left(u^+ + \varepsilon\right)^2} \leq C < \infty.$$

In particular, since $\left| \nabla_k u^+ \right|^2 \geq \left| \partial_1 u^+ \right|^2$ and thus

$$\int_\Omega \left| \partial_1 \ln\left(1 + \frac{u^+}{\varepsilon}\right) \right|^2 \leq C < \infty,$$

uniformly in $\varepsilon > 0$. Since u is continuous in $\overline{\Omega}$ and $u \leq 0$ on $\partial\Omega$, it then follows that $u \leq 0$ (a.e.) in Ω. Hence $w_1 \leq w_0$ in Ω as wanted.

References

[1] M. Christ, Hypoellipticity in the infinitely degenerate regime, preprint on website.

[2] V. S. Fediĭ, On a criterion for hypoellipticity, *Math. USSR Sbornik* **14** (1971), 15-45.

[3] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, revised 3rd printing, 1998.

[4] S. Kusuoka and D. Stroock, Applications of the Malliavin Calculus II, J. Fac. Sci. Univ. Tokyo **32** (1985), 1–76.

[5] J. Moser, On Harnack's theorem for elliptic differential equations, *Comm. Pure Appl. Math.* **14** (1961), 577-591.

[6] C. Rios, E. Sawyer and R. Wheeden, A higher dimensional partial Legendre transform, and regularity of degenerate Monge-Ampère equations, *preprint.*

[7] E. Sawyer, A symbolic calculus for rough pseudodifferential operators, *Indiana U. Math. J.* **45** (1996), 289-332.

[8] E. Sawyer and R. Wheeden, Regularity of degenerate Monge-Ampère and prescribed Gaussian curvature equations in two dimensions, *preprint.*

[9] F. Treves, Pseudodifferential and Fourier Integral Operators, Volume 1, The Univ. Ser. in Math., New York and London, 1980.

McMaster University, Hamilton, Ontario

McMaster University, Hamilton, Ontario

Rutgers University, New Brunswick, N.J.