Methods of Point Estimation

- Moments
- Maximum Likelihood

Method of Moments

- Estimate the mean (first moment) and variance (second moment)
- Use relationship between moments and parameter to estimate parameter

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i; \qquad s^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Example:

Hydraulic conductivity data for a sandy soil are given. It is suggested that a log-normal distribution will fit the shape of the data. Estimate the parameters λ and ζ of the log-normal distribution.

Solution: $\lambda = E(\ln X)$ $\zeta = \sqrt{Var(\ln X)}$ 4-2

$$f(x) = \frac{1}{\sqrt{2\pi}\,\zeta x} \exp\left[-\frac{1}{2}\frac{(\ln x - \lambda)^2}{\zeta^2}\right] \quad 0 \le x < \infty$$

$$E(X) = \exp\left(\lambda + \tfrac{1}{2}\zeta^2\right)$$

$$Var(X) = \exp(2\lambda + \zeta^2)(e^{\zeta^2} - 1)$$

$$E(X^2) = \exp\left[2(\lambda + \zeta^2)\right]$$

$$A_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$A_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

let $A_1 = E(X)$

$A_2 = E(X^2)$

$\Rightarrow$ λ and ζ can be solved.

Example: Method of Maximum Likelihood

A triaxial specimen of saturated sand is subjected to cyclic vertical loads with a stress amplitude of ± 200 psf in a laboratory test. The number of load cycles applied until the sand specimen fails has been recorded for five independent specimens as follows:

25, 20, 28, 33, 26 cycles

Suppose the number of load cycles to failure for the sand is assumed to follow a log-normal distribution; estimate the parameters λ and ζ by the maximum likelihood method.

A random variable X has a logarithmic normal (or simply) log-normal probability distribution if ln X is normal.

$$f(x) = \frac{1}{\sqrt{2\pi}\,\zeta x} \exp\left[-\frac{1}{2}\frac{(\ln x - \lambda)^2}{\zeta^2}\right] \quad 0 \le x < \infty$$

where $\lambda = E(\ln X)$ $\zeta = \sqrt{Var(\ln X)}$

$$E(x) = \mu = \exp\left(\lambda + \tfrac{1}{2}\zeta^2\right)$$

$$Var(x) = \sigma^2 = \exp(2\lambda + \zeta^2)(e^{\zeta^2} - 1)$$

Solution:

$$L(x_1, x_2, \cdots x_n; \lambda, \zeta) =$$

$$\prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{2\pi}\, x_i \zeta} \exp\left[-\frac{1}{2}\frac{(\ln x_i - \lambda)^2}{\zeta^2}\right]\right\}$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\zeta}\right)^n \prod_{i=1}^{n} x_i^{-1} \exp\left[-\frac{1}{2\zeta^2}\sum_{i=1}^{n}(\ln x_i - \lambda)^2\right]$$

$$\ln L(x_1, x_2, \cdots, x_n; \lambda, \zeta) =$$

$$-n \ln \sqrt{2\pi} - n \ln \zeta - \sum_{i=1}^{n} \ln x_i - \frac{1}{2\zeta^2}\sum_{i=1}^{n}(\ln x_i - \lambda)^2$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{1}{\zeta^2}\sum_{i=1}^{n}(\ln x_i - \lambda) = 0$$

$$\frac{\partial \ln L}{\partial \zeta} = -\frac{n}{\zeta} + \frac{1}{\zeta^3}\sum_{i=1}^{n}(\ln x_i - \lambda)^2 = 0$$

Since $\varsigma \neq 0$. We get

$$\begin{cases} \hat{\lambda} = \dfrac{\sum_{i=1}^{n} \ln x_i}{n} \\ \varsigma^2 = \dfrac{1}{n} \sum_{i=1}^{n} (\ln x_i - \lambda)^2 \end{cases}$$

$$\tilde{\lambda} = \frac{1}{5}(\ln 25 + \ln 20 + \ln 28 + \ln 33 + \ln 26) = 3.26$$

$$\varsigma^2 = \frac{1}{5}\left[(\ln 25 - 3.26)^2 + (\ln 20 - 3.26)^2 + (\ln 28 - 3.26)^2 + (\ln 33 - 3.26)^2 + (\ln 26 - 3.26)^2\right]$$

$$= 0.027$$

$$\varsigma = 0.1643.$$