Chapter 2
Probability

EXAMPLE 1:

The waste from an industrial plant is subjected to treatment before it is ejected to a nearby stream. The treatment process consists of three stages, namely: primary, secondary, and tertiary treatments. The primary treatment may be rated as good (G_1), incomplete (I_1) or failure (F_1). The secondary treatment may be rated as good (G_2) or failure (F_2), and the tertiary treatment may also be rated as good (G_3) or failure (F_3). Assume that the ratings in each treatment are equally likely (for example, the primary treatment will be equally likely to be good or incomplete or failure). Furthermore, the performances of the three stages of treatment are statistically independent of one another.

(a) What are the possible combined ratings of the three treatment stages? (for example, G_1, F_2, G_3 denotes a combination where there is a good primary and tertiary, but a failure in the secondary treatment). What is the probability of each of these combinations (or sample points)?

(b) suppose the event of satisfactory overall treatment requires at least two stages of good treatment. What is the probability of this event?

(c) Suppose:

E_1 = good primary treatment

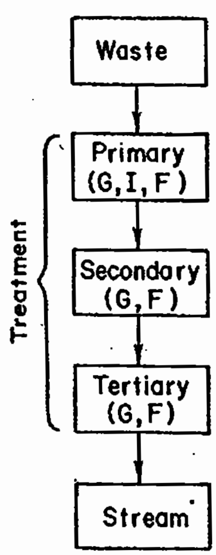
E_2 = good secondary treatment

E_3 = good tertiary treatment

Determine:

$$P\left(\bar{E}_1\right), \quad P\left(E_1 \cup E_2\right), \quad P\left(E_2 E_3\right)$$

(d) Express in terms of E_1, E_2, E_3 the event of satisfactory overall treatment as defined in part (b).



a) $G_1\ G_2\ G_3 = (1/3)(1/2)(1/2) = 1/12$ (statistically independent)

$G_1\ G_2\ F_3 = 1/12$

$G_1\ F_2\ G_3 = 1/12$

$G_1\ F_2\ F_3 = 1/12$

$I_1\ G_2\ G_3 = 1/12$

$I_1\ G_2\ F_3 = 1/12$

$I_1\ F_2\ G_3 = 1/12$

$I_1\ F_2\ F_3 = 1/12$

$F_1\ G_2\ G_3 = 1/12$

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$F_1\ F_2\ F_3 = 1/12$

b) Let S = satisfactory overall treatment

5 sample points $(G_1 G_2 G_3)$ $(G_1 G_2 F_3)$ $(G_1 F_2 G_3)$ $(I_1 G_2 G_3)$ $(F_1 G_2 G_3)$

$P(S) = \frac{5}{12}$ (mutually exclusive)

c) $P(\bar{E}_1) = 1 - P(E_1) = 1 - P(G_1) = 1 - \frac{1}{3} = \frac{2}{3}$

$$\begin{aligned} P(E_1 \cup E_2) &= P(E_1) + P(E_2) - P(E_1 E_2) \\ &= P(G_1) + P(G_2) - P(G_1 G_2) \\ &= 1/3 + 1/2 - (1/3)(1/2) = \frac{2}{3} \end{aligned}$$

$P(E_2 E_3) = P(G_2) \cdot P(G_3) = 1/2 \cdot 1/2 = 1/4$

d) $S = E_1E_2 \cup E_1E_3 \cup E_2E_3 \cup E_1E_2E_3$

EXAMPLE 2:

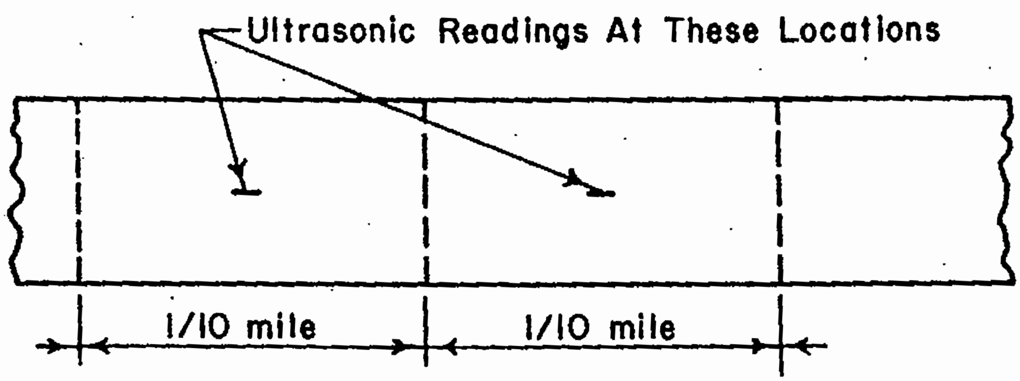
Before a section (say 1/10 mile long) of a pavement is accepted by the Ministry of Transportation, the thickness of an 8-in. pavement is inspected for specification compliance by ultrasonics reading. This is done at every 1/10-mile point of the pavement; each 1/10-mile section will be accepted if the measured thick-ness is at least 7.5 in.; otherwise the entire section will be rejected.

Suppose, from past experience that 90% of all sections constructed by the contractor were found to be in compliance with specifications. However, the ultrasonics thickness determination is only 80% reliable; that is, there is a 20% chance that a conclusion based on ultrasonics test may be erroneous.

(a) What is the probability that a particular section of the pavement is well constructed (that is, at least 7.5-in thick) and will be accepted by the Ministry of Transportation?

(b) What is the probability that a section is poorly constructed (that is, has thickness less than 7.5-in) but will be accepted on the basis of the ultrasonics test?

(c) What is the probability that if a section is well constructed, it will be accepted on these basis of the ultrasonics test?



Let G = actual thickness of pavement is at least 7.5"

A = measured thickness $\geq$ 7.5 in.

"reliability of ultrasonics test is 80%" may be interpreted to mean

$$P(G/A) = .8 \quad \text{and} \quad P(\bar{G} \backslash \bar{A}) = 0.8$$

hence $P(\bar{G}/A) = 0.2$

Based on the contractor's past record, we may assume that 90% of his work will have satisfactory ultrasonics reading

$$P(A) = 0.9$$

a) The event of interest is GA

$$P(GA) = P(G/A)\,P(A)$$ (statistically dependent)

$$= (0.8)(0.9) = 0.72$$

b) $$P(\bar{G}A) = P(\bar{G}/A)\,P(A)$$

$$= (0.2)(0.9) = 0.18$$

c) $$P(A|G) = \frac{P(G|A)\,P(A)}{P(G)} \quad \text{(Baye's theorem)}$$

$P(G|A) = 0.8$ and $P(A) = 0.9$)

To determine $P(G)$ we observe that A and $\bar{A}$ are mutually exclusive and collectively exhaustive

$$P(G) = P(G|A)P(A) + P(G|\bar{A})P(\bar{A})$$

$$= (0.8)(0.9) + (0.2)(0.1) \quad \text{(Total probability formula)}$$

$$= 0.74$$

$$P(A|G) = \frac{(0.8)(0.9)}{(0.74)} = 0.973$$

$$P(\bar{A}/G) = 1 - 0.973 = 0.027$$

Probability that a well constructed section may be rejected on the basis of the ultrasonics test.

EXAMPLE 3:

The air pollution in a city is caused mainly by industrial and automobile exhausts. In the next 5 years, the chances of successfully controlling these two sources of pollution are, respectively, 75% and 60%. Assume that if only one of the two sources is successfully controlled, the probability of bringing the pollution below acceptable level sould be 80%.

(a) What is the probability of successfully controlling air pollution in the next 5 years?

(b) If, in the next 5 years, the pollution level is not sufficiently controlled, what is the probability that it is entirely caused by the failure to control automobile exhaust?

(c) If pollution is not controlled, what is the probability that control of automobile exhaust was not successful?

Assuming statistical independence between controlling industrial (I) and automobile (A) exhausts, we have

a) $P(AI) = 0.75 \times 0.6 = 0.45$

$P(A\bar{I}) = 0.25 \times 0.6) = 0.15$

$P(\bar{A}I) = 0.75 \times 0.4 = 0.3$

$P(\bar{A}\bar{I}) = 0.25 \times 0.4 = 0.1$

Let E = event that air pollution is controlled

$$P(E) = 1(0.45) + 0.8(0.15) + 0.8(0.3) + 0(0.1)$$
$$= 0.81$$

b) $$P(\bar{A}I|\bar{E}) = \frac{P(\bar{E}|\bar{A}I)\,P(\bar{A}I)}{P(\bar{E})} = \frac{(0.2)(0.3)}{0.19} = 0.32$$

c) $$P(\bar{A}/\bar{E}) = P(\bar{A}I \cup \bar{A}\bar{I}|\bar{E})$$
$$= P(\bar{A}I/\bar{E}) + P(\bar{A}\bar{I}/\bar{E})$$
$$= \frac{P(\bar{E}/\bar{A}I)\,P(\bar{A}I)}{P(\bar{E})} + \frac{P(\bar{E}|\bar{A}\bar{I})\,P(\bar{A}\bar{I})}{P(\bar{E})}$$
$$= \frac{(0.2)(0.3)}{0.19} + \frac{(1)(0.1)}{0.19} = 0.84$$

whereas $P(\bar{I}/\bar{E}) = P(\bar{I}A \cup \bar{I}\bar{A} \ / \bar{E})$

$$= P(\bar{I}A/\bar{E}) + P(\bar{I}\bar{A}/\bar{E})$$

$$= \frac{P(\bar{E}/\bar{I}A)\,P(\bar{I}A)}{P(\bar{E})} + \frac{P(\bar{E}/\bar{I}\bar{A})\,P(\bar{I}\bar{A})}{P(\bar{E})}$$

$$= \frac{(0.2)(0.15) + (1)(0.1)}{0.19} = 0.68$$