STATS 3N03/3J04

2004-10-13

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POISSON DISTRIBUTION

$$f(x) = e^{-\mu}\frac{\mu^x}{x!} \qquad x = 0, 1, 2, \cdots$$

BASED ON EXPONENTIAL SERIES:

$$\sum_{x=0}^{\infty} f(x) = e^{-\mu}\left\{1 + \mu + \frac{\mu^2}{2!} + \cdots\right\}$$

$$= e^{-\mu} e^{\mu} = 1$$

MEAN:

$$\sum_{x=0}^{\infty} x f(x) = \sum_{x=0}^{\infty} x\, e^{-\mu}\frac{\mu^x}{x!}$$

$$= \sum_{x=1}^{\infty} x\, e^{-\mu}\frac{\mu^x}{x!} = \mu\sum_{x=1}^{\infty} e^{-\mu}\frac{\mu^{x-1}}{(x-1)!}$$

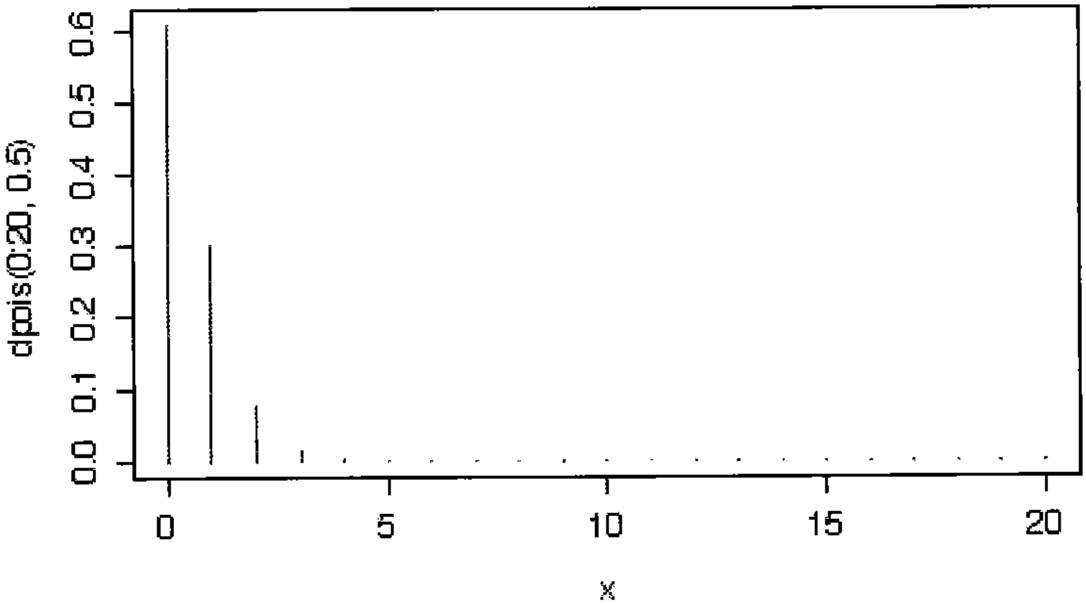
$$= \mu\left\{e^{-\mu}\left(1 + \mu + \frac{\mu^2}{2!} + \cdots\right)\right\}$$

$$= \mu$$

VARIANCE:

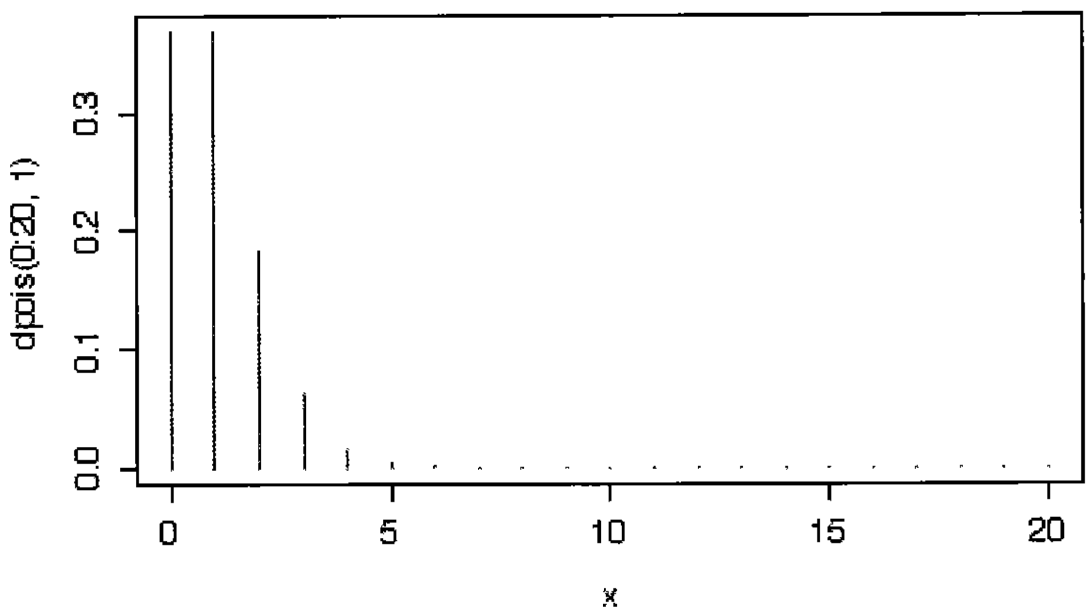
$$\sum_{x=0}^{\infty}(x-\mu)^2 f(x) = \cdots = \mu$$

```
> plot(0:20, dpois(0:20, .5), type="h", col="blue", xlab="x")
```



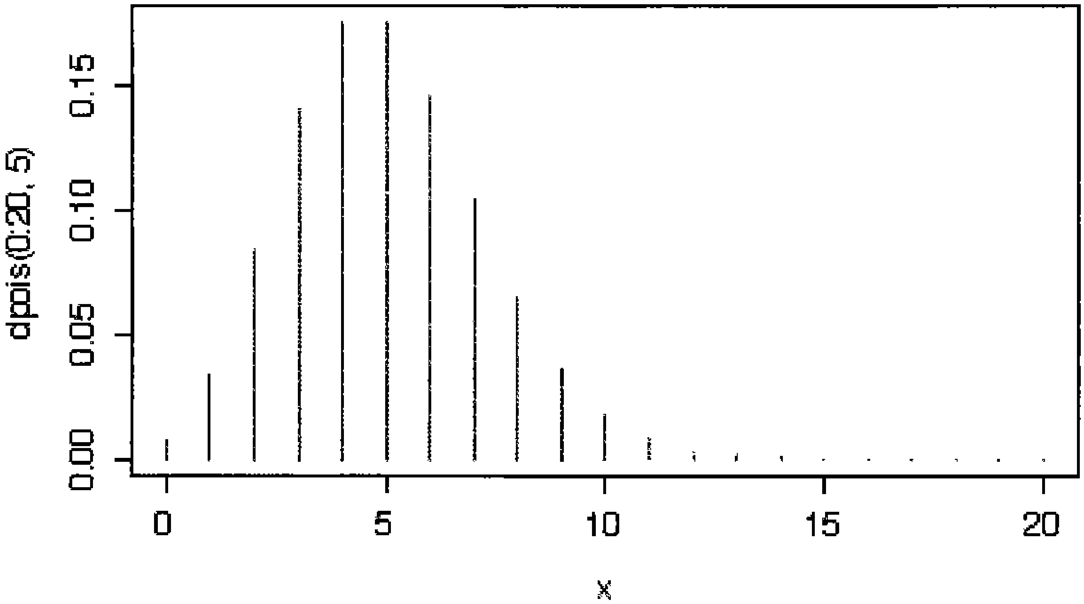
$\mu < 1$

```
> plot(0:20, dpois(0:20, 1), type="h", col="blue", xlab="x")
```



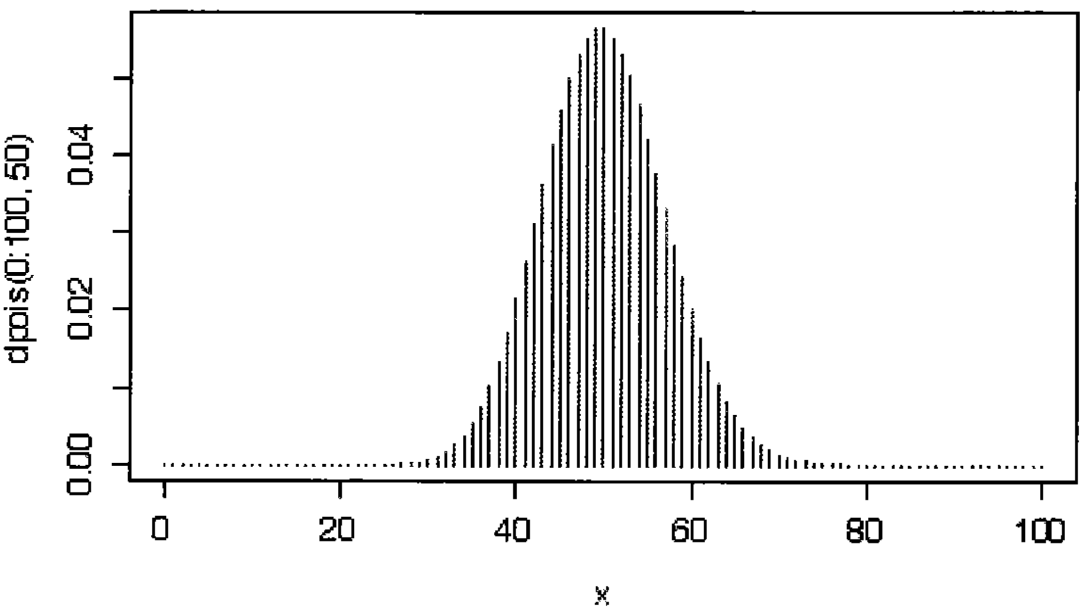
$\mu = 1$

```
> plot(0:20, dpois(0:20, 5), type="h", col="blue", xlab="x")
```



$\mu > 1$

```
> plot(0:100, dpois(0:100, 50), type="h", col="blue", xlab="x")
```



POISSON'S BINOMIAL LIMIT:

$$\lim_{\substack{n \to \infty \\ p \to 0 \\ np = \mu}} \binom{n}{x} p^x (1-p)^{n-x} = e^{-\mu} \frac{\mu^x}{x!}$$

$\Rightarrow$ WHEN n IS LARGE AND p IS SMALL, CAN APPROXIMATE THE BINOMIAL BY THE POISSON.

NOTE: $\lim_{\substack{n \to \infty \\ p \to 0 \\ np = \mu}} np(1-p) = \mu$

POISSON VARIANCE

BINOMIAL VARIANCE

EX PRODUCTION LINE GIVES 1% DEFECTIVE; WHAT IS THE PROBABILITY OF 2 OR MORE DEFECTIVES IN A LOT OF 100 ITEMS? STATE ANY ASSUMPTIONS.

LET X BE NO. DEFECTIVE IN A RANDOMLY CHOSEN LOT. ASSUME ITEMS ARE DEFECTIVE INDEPENDENTLY OF EACH OTHER. THEN

$$X \sim \text{Bin}(100, 0.01)$$

$$P(X \geq 2) = 1 - P(X \leq 1)$$

$$= 1 - \{P(X=0) + P(X=1)\}$$

$$= 1 - \left\{\binom{100}{0}(.01)^0(.99)^{100} + \binom{100}{1}(.01)^1(.99)^{99}\right\}$$

$$= 1 - .99^{100} - (100)(.01)(.99)^{99}$$

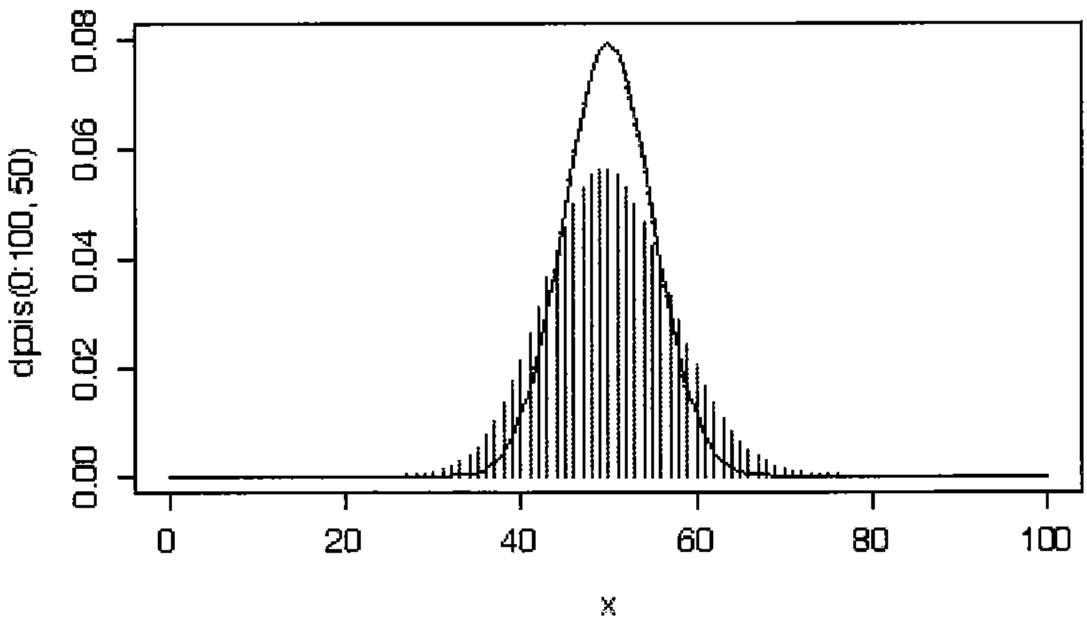
$$= 1 - 0.366 - 0.370 = 0.264$$

POISSON APPROXIMATION: $\mu = 1$

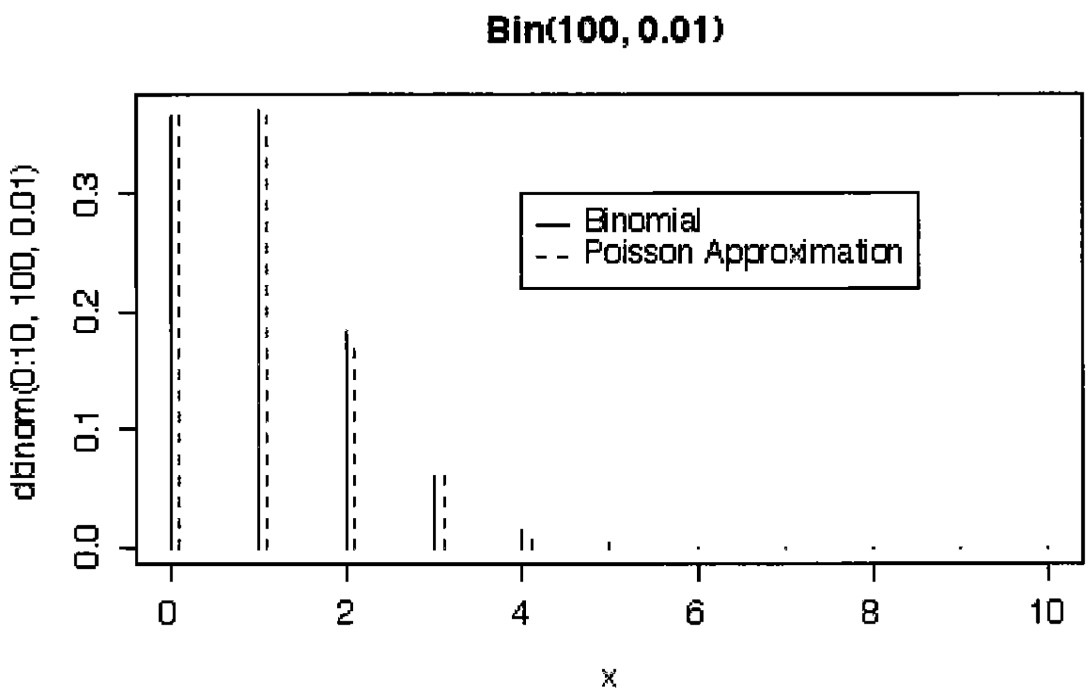
$$P(X \geq 2) \approx 1 - \{P_{\text{Pois}}(X=0) + P_{\text{Pois}}(X=1)\}$$

$$= 1 - \{e^{-1} + e^{-1}\tfrac{1}{1!}\} = 1 - 2e^{-1} = 0.264$$

```
> lines(0:100, dbinom(0:100, 100, .5), col="red")
```




```
> plot(0:10, dbinom(0:10, 100, .01), type="h", col="red",
xlab="x")
> lines((0:10)+.1, dpois(0:10, 1), type="h", lty=2, col="blue")
> legend(4, .3, c("Binomial", "Poisson Approximation"),
lty=c(1,2), col=c("red", "blue"))
> title(main="Bin(100, 0.01)")
```



POISSON PROCESS

- STREAM OF POINT EVENTS IN CONTINUOUS TIME OR SPACE
- EVENTS HAPPEN ONE AT A TIME, INDEPENDENTLY OF EACH OTHER, AT A CONSTANT AVERAGE RATE ν

LET X_t = NUMBER OF EVENTS IN $(0, t]$

RESULT: $X_t \sim \mathrm{POIS}(\nu t)$

EX CABLE SHIPPED IN 100 m ROLLS

FLAWS: $0.0025 \, m^{-1}$

ASSUME: FLAWS INDEPENDENT OF EACH OTHER, CONSTANT AVERAGE RATE, ONE AT A TIME.

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LET X = NUMBER OF FLAWS IN A ROLL.

$$\nu t = (.0025)(100) = 0.25$$

$$\Rightarrow X \sim \text{POIS}(0.25)$$

IF YOU BUY 5 ROLLS

LET Y = NUMBER OF ROLLS WITH AT LEAST 1 FLAW

$$\Rightarrow Y \sim \text{BN}(5, p)$$

WHERE

$$p = P(X \geq 1) = 1 - P(X=0)$$

$$= 1 - e^{-0.25} \frac{(0.25)^0}{0!}$$

$$= 0.2212$$

[NOTE μ SMALL $\Rightarrow 1 - e^{-\mu} \approx \mu$]

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P(NO FLAWED ROLLS)

$= P(Y=0) = \binom{5}{0} p^0 (1-p)^{5-0}$

$= (e^{-0.25})^5 = e^{-1.25} = 0.2865$

WHAT IS THE PROBABILITY THAT AT MOST 2 ROLLS HAVE 2 OR MORE FLAWS EACH?