Some nice formulas

1. Definitions:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt = 2\int_0^\infty t^{2z-1}e^{-t^2}\,dt = \int_0^1 (-\log(t))^{z-1}\,dt \qquad \text{for } Re(z) > 0$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = 2\int_0^{\frac{\pi}{2}} sin^{2x-1}\theta\, cos^{2y-1}\theta\, d\theta$$

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_{\text{all primes p}} (1-p^{-s})^{-1} \qquad \text{for } Re(s) > 1$$

$$\eta(s) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^s}\,dx \qquad \text{for } Re(s) > 0$$

$$\zeta(s) = \frac{1}{1-2^{1-s}}\,\eta(s) \qquad \text{for } Re(s) > 0$$

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \qquad \text{for } Re(s) > 0$$

These functions have meromorphic (only poles no essential singularities) extensions to the whole complex plane $\mathbb{C}$. This is achieved by contour integrals and functional identities.

2. Identities:

$$\Gamma(z+1) = z\Gamma(z)$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\,dt$$

$$\Gamma(s)\eta(s) = \int_0^\infty \frac{t^{s-1}}{e^t + 1}\,dt$$

$$\xi(s) = \xi(1-s)$$

3. Special values

$$\Gamma(n) = n! \;\text{ for } n \in \mathbb{N} \qquad \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\zeta(2) = \frac{\pi^2}{6} \quad \zeta(4) = \frac{\pi^4}{90} \qquad \zeta(2n) = (-1)^n B_{2n}\frac{2^{2n-1}\pi^{2n}}{(2n)!} \qquad \text{for } n \in \mathbb{N}$$

$$\zeta(0) = -\frac{1}{2} \quad \zeta(-1) = -\frac{1}{12} \qquad \zeta(-2n+1) = -\frac{B_{2n}}{2n} \qquad \zeta(-2n) = 0 \qquad \text{for } n \in \mathbb{N}$$

$$\zeta'(0) = -\frac{1}{2}\log(2\pi) \qquad \prod_{n=1}^\infty n = \exp(-\zeta'(0)) = \sqrt{2\pi}$$

The B_n are the Bernoulli numbers which show up in the following Taylor/Laurent expansions:

4. Series expansions:

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}$$

$$\cot z = \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^k B_{2k} \frac{(2)^{2k}}{(2k)!} z^{2k-1}$$

$$\csc(z) = \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^{k-1} B_{2k} \frac{2(2^{2k-1} - 1)}{(2k)!} z^{2k-1}$$