

A LAW OF LARGE NUMBERS FOR THE ZEROES OF HEINE-STIELTJES POLYNOMIALS

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ABSTRACT. We determine the limiting density of the zeroes of Heine-Stieltjes polynomials in the thermodynamic limit and use this to prove a strong law of large numbers for the zeroes.

1. INTRODUCTION

In this paper, we discuss the asymptotic distribution of the zeroes of the polynomial solutions, ϕ_K , of degree K of the Heine-Stieltjes equation [Sz]:

$$\frac{d^2}{dx^2}\phi_K(x) + 2\left(\sum_{\nu=0}^N \frac{\beta_\nu}{x - \alpha_\nu}\right) \frac{d}{dx}\phi_K(x) = \frac{C(x)}{A(x)}\phi_K(x), \quad (1)$$

where, $0 < \alpha_0 < \alpha_1 < \dots < \alpha_N$, $\beta_\nu > 0$ for all $\nu = 0, \dots, N$, $C(x)$ is a polynomial of degree $N - 1$ and $A(x) = (x - \alpha_0) \cdots (x - \alpha_N)$. The polynomial $C(x)$ is called a *Van Vleck* polynomial. Such ODE's arise naturally when separating variables in certain (quantum) completely integrable systems. More precisely, consider the generalized (real) Gaudin spin chains [HW, K, KKN, KM, Sk]. These consist of N explicit pairwise-commuting, second-order elliptic differential operators $P_1^\alpha, \dots, P_N^\alpha$ acting on $C^\infty(\mathbb{S}^N)$. In terms of a suitable (elliptic) coordinate system u_1, \dots, u_N on the sphere, the joint eigenfunctions, $\psi_K(u_1, \dots, u_N)$ of the P_j^α 's are of product form $\psi_K(u_1, \dots, u_N) = \prod_{j=1}^N \phi_K(u_j)$, where the ϕ_K 's are precisely the polynomial solutions to (1) (ie. the Heine-Stieltjes polynomials). Moreover, the coefficients of the polynomial $C(x)$ are the joint eigenvalues of the P_j^α 's with joint eigenfunction, ψ_K . For background, we refer to [T1, T2, Sz, WW]. It is a natural question to determine the distribution of the zeroes of ϕ_K in various asymptotic regimes. Recently, Martinez-Finkelstein and Saff [MS] have studied the density states of the Heine-Stieltjes zeroes as $K \rightarrow \infty$ (ie. semiclassical asymptotics).

In this paper, we study the asymptotics of the density of states of the zeroes of the Heine-Stieltjes polynomials as the dimensional parameter $N \rightarrow \infty$. More precisely, let $\Omega(K, N)$ denote the finite set of monic Heine-Stieltjes polynomials of degree K depending on N "spin sites" $0 < \alpha_0 < \dots < \alpha_N < 1$. Then, it is well-known (see [Sz] and Theorem 2.1) that

$$\text{card}(\Omega(K, N)) = \sigma(K, N) = \frac{(N + K - 1)!}{K!(N - 1)!}.$$

We consider the density of states measure

$$d\rho_{DS}(x; m, K, N, \alpha) := \frac{1}{K} \sum_{j=1}^K \delta(x - \theta_j^{(m)}(\alpha)), \quad (2)$$

where, $\theta_j^{(m)}$ denotes the j th zero of the m th polynomial in the set $\Omega(K, N)$. We define the averaged density of states in a similar fashion:

$$d\rho_{DS}^{AV}(x; K, N, \alpha) := \frac{1}{\sigma(K, N)} \sum_{m=1}^{\sigma(K, N)} \frac{1}{K} \sum_{j=1}^K \delta(x - \theta_j^{(m)}(\alpha)). \quad (3)$$

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Sections 3 through 7 are devoted to the study of the large N asymptotics of the zeroes of $d\rho_{DS}^{AV}(x; m, K, N, \alpha)$ and $d\rho_{DS}(x; m, N, K, \alpha)$.

Our first result, Proposition 3.1, concerns the asymptotics of the averaged density of states: We show that for any $\phi \in BV([0, 1])$ and any $\alpha \in \Lambda^N := \{\alpha \in [0, 1]^{N+1} : 0 < \alpha_0 < \dots < \alpha_N < 1\}$,

$$d\rho_{DS}^{AV}(\phi)(\alpha) = \frac{1}{N+1} \sum_{\nu=0}^N \phi(\alpha_\nu) + \mathcal{O}\left(\frac{1}{N}\right), \quad (4)$$

as $N \rightarrow \infty$. In (4), we use the shorter notation $d\rho_{DS}^{AV}(\phi)(\alpha)$ to denote $d\rho_{DS}^{AV}(\phi; K, N, \alpha)$. We will use this notation throughout the rest of the text.

In Theorem 5.1, we strengthen this result by giving the following estimate for the variance of $d\rho_{DS}(x; m, K, N, \alpha)$ itself:

$$\frac{1}{\sigma(K, N)} \sum_{m=1}^{\sigma(K, N)} \left[d\rho_{DS}(\phi)(\alpha) - \frac{1}{N+1} \sum_{\nu=0}^N \phi(\alpha_\nu) \right]^2 = \mathcal{O}\left(\frac{1}{K}\right) + \mathcal{O}\left(\frac{1}{N}\right). \quad (5)$$

Thus, as long as the degree K and the number of spin sites N *both* go to infinity, the typical Heine-Stieltjes polynomial has its zeroes distributed uniformly in the interval $[0, 1]$. As a corollary to Theorem 5, we arrive at the main result of the paper (Theorem 7.1), a strong law of large numbers for the zeroes of generic Heine-Stieltjes polynomials.

Remarks: (i) It would be interesting to determine whether the zeroes of the joint eigenfunctions, ψ_K , have an invariant asymptotic distribution on \mathbb{S}^N itself relative to the standard round metric. We hope to address this elsewhere.

(ii) We were motivated by the previous work of two of the authors [BT] on the averaged level-spacings of the Heine-Stieltjes zeroes. There are several significant improvements here that we wish point out: (1) By a simple combinatorial monotonicity argument, we get *pointwise* results in the spin sites, $\alpha_0, \dots, \alpha_N$ and completely avoid averaging over this parameter space. (2) We prove a result about the asymptotic density of states of a ‘‘typical’’ Heine-Stieltjes polynomial (Theorem 5.1); that is, about $d\rho_{DS}$ itself and not just $d\rho_{DS}^{AV}$. (3) We need not assume anything about the relative size of the degree, K , and the dimension, N , in any of our results.

(iii) The techniques of the present paper allow us to give an alternative proof of the results in [B]. We hope to address this in future work.

2. THE HEINE-STIELTJES THEOREM

The combinatorial component of our analysis is based in large part on the following classical result of Heine and Stieltjes [H, St, Sz]:

Theorem 2.1. (*Heine-Stieltjes*) *Let $A(x)$ be the polynomial of degree $N + 1$ given by*

$$A(x) = (x - \alpha_0) \cdot (x - \alpha_1) \cdots (x - \alpha_N),$$

where $0 < \alpha_0 < \alpha_1 < \dots < \alpha_N$ and $B(x)$ a polynomial of degree N satisfying the condition

$$\frac{B(x)}{A(x)} = \frac{\beta_0}{x - \alpha_0} + \dots + \frac{\beta_N}{x - \alpha_N},$$

for given numbers $\beta_\nu > 0$, $\nu = 0, \dots, N$. Then, there are exactly $\sigma(K, N) = \frac{(N+K-1)!}{K!(N-1)!}$ polynomials $C(x)$ of degree $N - 1$ for which the differential equation

$$A(x) \frac{d^2\phi}{dx^2} + 2B(x) \frac{d\phi}{dx} + C(x) \phi = 0 \quad (6)$$

has a polynomial solution of degree $K > 0$. In addition, for each of the $\sigma(K, N)$ solutions, $\phi(x)$, the zeroes are simple and uniquely determined by their distribution in the intervals $(\alpha_0, \alpha_1), \dots, (\alpha_{N-1}, \alpha_N)$.

Taking into account the Heine-Stieltjes result, we henceforth denote the zeroes of any Heine-Stieltjes polynomials $\phi_K(x) \in \Omega(K, N)$ by $\theta_1(\alpha; \underline{l}), \dots, \theta_K(\alpha; \underline{l})$, where $\alpha := (\alpha_0, \dots, \alpha_N)$ whereas $\underline{l} = (l_1, \dots, l_K)$, $1 \leq l_1 \leq \dots \leq l_K \leq N$ denotes the configuration of the zeroes. By this we mean that $\theta_1(\alpha; \underline{l})$ is the smallest zero lying in the interval $(\alpha_{l_1-1}, \alpha_{l_1})$, the next zero $\theta_2(\alpha; \underline{l})$ is contained in the interval $(\alpha_{l_2-1}, \alpha_{l_2})$ and so on.

3. MEAN DENSITY OF STATES FOR THE HEINE-STIELTJES POLYNOMIALS

Let $\phi \in BV([0, 1])$ with total variation $V(\phi)$. As a consequence of the notation introduced in Section 2, we can rewrite Eq. (3) in the form:

$$d\rho_{DS}^{AV}(\phi)(\alpha) = \frac{1}{\sigma(K, N)} \sum_{|\underline{l}|=K} \frac{1}{K} \sum_{j=1}^K \phi(\theta_j(\alpha; \underline{l})). \quad (7)$$

Here, we use the notation $|\underline{l}| = K$ to denote all K -tuples $\underline{l} = (l_1, \dots, l_K)$ of positive integers satisfying the conditions $1 \leq l_1 \leq \dots \leq l_K \leq N$. Our first result is about the asymptotics of the averaged measure $d\rho_{AV}^{DS}$:

Proposition 3.1. *Let $\phi \in BV([0, 1])$ and choose any sequence $K = K(N)$ with $N \rightarrow \infty$. Then,*

$$d\rho_{DS}^{AV}(\phi)(\alpha) = \frac{1}{N+1} \sum_{j=0}^N \phi(\alpha_j) + E(N),$$

where,

$$|E(N)| \leq \frac{2V(\phi)}{N}.$$

Proof of Proposition 3.1: The first step is to write $\phi = \phi_+ - \phi_-$ where $\phi_{\pm} \in BV([0, 1])$ are monotone, non-decreasing functions. Split the expression

$$d\rho_{DS}^{AV}(\phi)(\alpha) = d\rho_{DS}^{AV}(\phi_+)(\alpha) - d\rho_{DS}^{AV}(\phi_-)(\alpha). \quad (8)$$

We do the computations for $d\rho_{AV}^{DS}(\phi_+; K, N)$ and $d\rho_{AV}^{DS}(\phi_-; K, N)$ separately. We call α_j interior if $j \neq 0, N$. The points α_0 and α_N are boundary points. Then,

$$d\rho_{AV}^{DS}(\phi_{\pm})(\alpha) = \frac{1}{\sigma(K, N)} \sum_{|\underline{l}|=K} \frac{1}{K} \sum_{j=1}^K \phi_{\pm}(\alpha_{l_j-1}) + E(N), \quad (9)$$

where,

$$\begin{aligned} E(N) &= \frac{1}{\sigma(K, N)} \sum_{|\underline{l}|=K} \frac{1}{K} \sum_{j=1}^K [\phi_{\pm}(\theta_j(\alpha; \underline{l})) - \phi_{\pm}(\alpha_{l_j-1})] \\ &\leq \frac{1}{\sigma(K, N)} \sum_{|\underline{l}|=K} \frac{1}{K} \sum_{j=1}^K [\phi_{\pm}(\alpha_{l_j}) - \phi_{\pm}(\alpha_{l_j-1})]. \end{aligned} \quad (10)$$

Here, the line (10) follows from the monotonicity of ϕ_{\pm} and the inequalities,

$$\alpha_{l_j-1} < \theta_j(\alpha; \underline{l}) < \alpha_{l_j},$$

that are consequences of the Heine-Stieltjes Theorem.

Next, we claim that

$$\frac{1}{\sigma(K, N)} \sum_{|\underline{l}|=K} \frac{1}{K} \sum_{j=1}^K [\phi_{\pm}(\alpha_{l_j}) - \phi_{\pm}(\alpha_{l_j-1})] = \frac{\text{card}(\{\underline{l}; l_1 = 1\})}{\sigma(K, N)} \phi_{\pm}(\alpha_0) - \frac{\text{card}(\{\underline{l}; l_K = N\})}{\sigma(K, N)} \phi_{\pm}(\alpha_N). \quad (11)$$

The identity in (11) is proved as follows: Let $m > K$ and fix any $m - K$ zeroes x_1, \dots, x_{m-K} with m zeroes $x_{m-K+1}, \dots, x_K \in (\alpha_j, \alpha_{j+1})$. Then by Heine-Stieltjes, for any such m we can choose a corresponding configuration with zeroes y_1, \dots, y_{m-K} in exactly the same relative position versus the α 's as x_1, \dots, x_{m-K}

and put $y_{m-K+1}, \dots, y_K \in (\alpha_{j-1}, \alpha_j)$. It follows that $\phi_{\pm}(\alpha_j)$ appears with the same number of plus and minus signs in (11) as long as α_j is an interior point. So, we are left with computing the total number of occurrences of $-\phi_{\pm}(\alpha_0)$ and $\phi_{\pm}(\alpha_N)$ in (11). This number is

$$\frac{\text{card}(\{l; l_1 = 1\})}{\sigma(K, N)} = \frac{\text{card}(\{l; l_K = N\})}{\sigma(K, N)} = \frac{\sum_{m=0}^K m \cdot \sigma(N-1, K-m)}{K \cdot \sigma(N, K)}. \quad (12)$$

The computation of the sum in the right-hand side of (12) follows from the following elementary counting lemma:

Lemma 3.2. *We have that*

$$\sum_{m=0}^K m \cdot \sigma(N-1, K-m) = \frac{K \cdot \sigma(N, K)}{N}.$$

Proof of Lemma 3.2: Switching two intervals $[\alpha_{i-1}, \alpha_i]$ and $[\alpha_{j-1}, \alpha_j]$ (i.e. a permutation of i and j) leaves a given configuration invariant iff the two intervals contain the same number of zeroes of the configuration. Otherwise it switches the two numbers. Therefore either a configuration has the same number of zeroes in the two intervals or it is mapped under the permutation to a different (unique) configuration having exactly the two numbers switched. Therefore if we take the union of all the zeroes of all the configurations each α interval contains exactly the same number of zeroes and since the total number of zeroes is $N\sigma(K, N)$ each of the N intervals has to contain exactly $M = \frac{K\sigma(K, N)}{N}$ zeroes. \square

Applying Lemma 3.2 together with Eqs. (8), (9), (10), (11) and (12) gives

$$d\rho_{DS}^{AV}(\phi)(\alpha) = \frac{1}{N+1} \sum_{\nu=0}^N \phi(\alpha_{\nu}) + E(N),$$

where,

$$|E(N)| \leq \frac{\phi_+(\alpha_N) - \phi_+(\alpha_0)}{N} + \frac{\phi_-(\alpha_N) - \phi_-(\alpha_0)}{N} \leq \frac{2V(\phi)}{N}.$$

This completes the proof of Proposition 3.1. \square

4. APPROXIMATION

We proved in Proposition 3.1 that as the number of α 's becomes arbitrary large as $(N \rightarrow \infty)$,

$$\frac{1}{\sigma(K, N)} \sum_{|\underline{l}|=K} \frac{1}{K} \sum_{j=1}^K \delta(\theta_j(\alpha; \underline{l})) \rightarrow \frac{1}{N+1} \sum_{i=0}^N \delta(\alpha_i),$$

with the error being $O(1/N)$. For brevity, we denote the measure $(1/K) \sum_{j=1}^K \delta(\theta_j(\alpha; \underline{l}))$ by $d\rho_{DS}(\underline{l})$ and the limiting measure $1/(N+1) \sum_{i=0}^N \delta(\alpha_i)$ by $d\mu_N$.

We now want to estimate the *variance*, over the space of all configurations of size K , of the difference between the two measures $d\rho_{DS}(\underline{l})$ and $d\mu_N$. We choose a test function $\phi \in \text{BV}([0, 1])$, and wish to estimate

$$\text{Var}(\phi) := \frac{1}{\sigma(K, N)} \sum_{|\underline{l}|=K} \left(\int \phi d\rho_{DS}(\underline{l}) - \int \phi d\mu_N \right)^2. \quad (13)$$

First, we rewrite the right-hand side of line (13) as

$$\sum_{|\underline{l}|=K} \left[\frac{1}{\sigma(K, N)K^2} \sum_{i,j=1}^K \phi(\theta_i(\alpha; \underline{l}))\phi(\theta_j(\alpha; \underline{l})) + \frac{2}{\sigma(K, N)K} \int \phi d\rho_{DS}(\underline{l}) \cdot \int \phi d\mu_N + \left(\int \phi d\mu_N \right)^2 \right].$$

We then use Proposition 3.1 for the sum of the second term in Eq. (14) to conclude that

$$\text{Var}(\phi) = \frac{1}{\sigma(K, N)K^2} \sum_{|\underline{l}|=K} \sum_{i,j=1}^K \phi(\theta_i(\alpha; \underline{l}))\phi(\theta_j(\alpha; \underline{l})) - \left(\frac{1}{N+1} \sum_{i=0}^N \phi(\alpha_i) \right)^2 + \mathcal{O}\left(\frac{1}{N}\right). \quad (14)$$

As for the proof of Proposition 3.1, we write $\phi = \phi_+ - \phi_-$ where ϕ_{\pm} are monotone increasing, and use the estimates

$$\phi_{\pm}(\alpha_{l_j-1}) \leq \phi_{\pm}(\theta_j(\alpha; \underline{l})) \leq \phi_{\pm}(\alpha_{l_j}), \quad (15)$$

that follows from the Heine-Stieltjes Theorem (see Section 2). We apply (15) to every term in the sum $\sum_{i,j=1}^K \phi_{\pm}(\theta_i(\alpha; \underline{l}))\phi_{\pm}(\theta_j(\alpha; \underline{l}))$ and add the resulting estimates for all $|\underline{l}| = K$. We get a lower bound for the sum $\sum_{|\underline{l}|=K} \sum_{i,j=1}^K \phi_{\pm}(\theta_i(\alpha; \underline{l}))\phi_{\pm}(\theta_j(\alpha; \underline{l}))$ of the form

$$L(\phi_{\pm}) := \sum_{i=0}^N L(i)(\phi_{\pm}(\alpha_i))^2 + \sum_{i<j} L(i, j)\phi_{\pm}(\alpha_i)\phi_{\pm}(\alpha_j), \quad (16)$$

where $L(i)$ denotes the number of times $\phi_{\pm}^2(\alpha_i)$ appears on the left-hand side of (15), and $L(i, j)$ denotes the number of times $\phi_{\pm}(\alpha_i)\phi_{\pm}(\alpha_j)$ appears on the left-hand side of (15), when added over all configurations.

Similarly, we get an upper bound for the sum $\sum_{|\underline{l}|=K} \sum_{i,j=1}^K \phi_{\pm}(\theta_i(\alpha; \underline{l}))\phi_{\pm}(\theta_j(\alpha; \underline{l}))$ of the form

$$U(\phi) := \sum_{i=0}^N U(i)(\phi_{\pm}(\alpha_i))^2 + \sum_{i<j} U(i, j)\phi_{\pm}(\alpha_i)\phi_{\pm}(\alpha_j), \quad (17)$$

where $U(i)$ denotes the number of times $\phi_{\pm}^2(\alpha_i)$ appears on the right-hand side of (15), and $L(i, j)$ denotes the number of times $\phi_{\pm}(\alpha_i)\phi_{\pm}(\alpha_j)$ appears on the right-hand side of (15), when added over all configurations.

We now proceed to calculate $L(i), U(i), L(i, j), U(i, j)$.

Proposition 4.1. *We have that*

- (a) $0 = U(0) = L(N)$.
- (b) $U(i) = L(j) = \sigma(N+2, K-1) + \sigma(N+2, K-2)$ for $i \neq 0, j \neq N$.
- (c) $0 = U(i, 0) = L(j, N)$, for all $i > 0$ and $j < N$.
- (d) $U(i, j) = L(i, j) = 2\sigma(N+2, K-2)$, for all $i < j$ not described in (c).

Proof of Proposition 4.1: The proof of (a) and (c) is clear: there are no θ -s to the right of α_N or to the left of α_0 .

Let $m_j(\underline{l})$ denote the number of zeroes occuring in the interval (α_{j-1}, α_j) ; $1 \leq j \leq N$ for the fixed configuration \underline{l} . The contribution to $U(i), 1 \leq i \leq N$ from this configuration is equal to $m_i(\underline{l})^2$. Thus,

$$U(i) = \sum_{|\underline{l}|=K} m_i(\underline{l})^2.$$

Similarly, the contribution from \underline{l} to $L(i), 0 \leq i \leq N-1$, is equal to $m_{i+1}(\underline{l})^2$, Thus,

$$L(i) = \sum_{|\underline{l}|=K} m_{i+1}(\underline{l})^2.$$

The statement of (b) now follows from Corollary 6.4.

We finally turn to the proof of (d). The contribution from a configuration \underline{l} to $U(i, j)$ is equal to $2m_i m_j$, $1 \leq i < j \leq N$. Accordingly,

$$U(i, j) = \sum_{|\underline{l}|=K} 2m_i(\underline{l})m_j(\underline{l}).$$

Similarly, the contribution from \underline{l} to $L(i, j)$, $0 \leq i < j \leq N-1$, is equal to $2m_{i+1}(\underline{l})m_{j+1}(\underline{l})$. Thus,

$$L(i, j) = 2 \sum_{|\underline{l}|=K} m_{i+1}(\underline{l})m_{j+1}(\underline{l}).$$

The statement of (d) now follows from Corollary 6.4. \square

We next show that $U(\phi_{\pm})$ and $L(\phi_{\pm})$ provide an asymptotically good approximation for the sum

$$\sum_{|\underline{l}|=K} \sum_{i,j=1}^K \phi_{\pm}(\theta_i(\alpha; \underline{l}))\phi_{\pm}(\theta_j(\alpha; \underline{l})).$$

Indeed,

Lemma 4.2. *Let $\Delta(\phi_{\pm}) := (U(\phi_{\pm}) - L(\phi_{\pm})) / (K^2\sigma(K, N))$. Then $\Delta(\phi_{\pm}) \rightarrow 0$ as $N \rightarrow \infty$.*

Proof of Lemma 4.2: From Eqs. (17), (16) and Proposition 4.1, we conclude that

$$\begin{aligned} \Delta(\phi_{\pm}) &= \frac{\sigma(N+2, K-1) + \sigma(N+2, K-2)}{K^2\sigma(K, N)} (\phi^2(\alpha_N) - \phi_{\pm}^2(\alpha_0)) \\ &\quad + \frac{2\sigma(N+2, K-2)(\phi_{\pm}(\alpha_N) - \phi_{\pm}(\alpha_0))}{K^2\sigma(K, N)} \cdot \sum_{i=1}^{N-1} \phi_{\pm}(\alpha_i). \end{aligned}$$

Since by definition $\sigma(K, N) = \binom{N+K-1}{N-1}$, we see that

$$\frac{\sigma(N+2, K-2)}{K^2\sigma(K, N)} = \frac{K-1}{KN(N+1)}, \quad \frac{\sigma(N+2, K-1)}{K^2\sigma(K, N)} = \frac{N+K}{KN(N+1)}. \quad (18)$$

It follows from line (18) and the fact that the functions ϕ_{\pm} are bounded, i.e. $|\phi_{\pm}(\alpha_{\nu})| \leq M$ for some fixed $M > 0$ and all ν ,

$$\Delta(\phi_{\pm}) \leq 2M^2 \left[\frac{N+2K-1}{KN(N+1)} + \frac{(K-1)N}{KN(N+1)} \right] = \mathcal{O}\left(\frac{2}{N^2} + \frac{1}{KN} + \frac{1}{N}\right) = \mathcal{O}\left(\frac{1}{N}\right).$$

This shows that $\Delta(\phi_{\pm}) \rightarrow 0$ as $N \rightarrow \infty$ and completes the proof of Lemma 4.2. \square

5. VARIANCE

As a consequence of Eq. (14) and Lemma 4.2, we see that

$$\text{Var}(\phi_{\pm}) = \frac{U(\phi_{\pm})}{K^2\sigma(K, N)} - \left(\frac{1}{N+1} \sum_{i=0}^N \phi_{\pm}(\alpha_i) \right)^2 + \mathcal{O}\left(\frac{1}{N}\right).$$

The last expression can be rewritten as

$$\text{Var}(\phi_{\pm}) = S_1 + S_2 + S_3 + \mathcal{O}(1/N),$$

where

$$S_1 := \sum_{i=1}^N \left(\frac{\sigma(N+2, K-2) + \sigma(N+2, K-1)}{K^2\sigma(K, N)} - \frac{1}{(N+1)^2} \right) \phi_{\pm}^2(\alpha_i),$$

where

$$S_2 := 2 \sum_{1 \leq i < j \leq N} \left(\frac{\sigma(N+2, K-2)}{K^2\sigma(K, N)} - \frac{1}{(N+1)^2} \right) \phi_{\pm}(\alpha_i)\phi_{\pm}(\alpha_j),$$

and where

$$-S_3 := \frac{\phi_{\pm}(\alpha_0)}{(N+1)^2} \sum_{j=0}^N \phi_{\pm}(\alpha_j).$$

It is clear that $S_3 = \mathcal{O}(1/N)$. We proceed to estimate S_1 and S_2 . Using (18), we see that

$$S_2 \leq \frac{2M^2N(N-1)}{2} \left(\frac{K-1}{KN(N+1)} - \frac{1}{(N+1)^2} \right) = \mathcal{O} \left(\frac{1}{K} + \frac{1}{N} \right).$$

Finally, using (18) once more, we see that

$$S_1 \leq NM^2 \left(\frac{N+2K-1}{KN(N+1)} - \frac{1}{(N+1)^2} \right) = \mathcal{O} \left(\frac{1}{K} + \frac{1}{N} \right).$$

Summing up, we have

$$\text{Var}(\phi_{\pm}) = \mathcal{O} \left(\frac{1}{K} \right) + \mathcal{O} \left(\frac{1}{N} \right). \quad (19)$$

We now come to the main theorem of this section.

Theorem 5.1. *Let $\phi \in BV([0, 1])$. Then, $\text{Var}(\phi) \rightarrow 0$ as $K \rightarrow \infty$ and $N \rightarrow \infty$. More precisely, we have*

$$\text{Var}(\phi) = \mathcal{O} \left(\frac{1}{K} \right) + \mathcal{O} \left(\frac{1}{N} \right).$$

Proof of Theorem 5.1: Since $\phi = \phi_+ - \phi_-$, it follows from Eq. (19) that

$$\begin{aligned} \text{Var}(\phi) &= \text{Var}(\phi_+) + \text{Var}(\phi_-) - 2\text{Cov}(\phi_+, \phi_-) \\ &= \mathcal{O} \left(\frac{1}{K} \right) + \mathcal{O} \left(\frac{1}{N} \right) - 2\text{Cov}(\phi_+, \phi_-), \end{aligned} \quad (20)$$

where $\text{Cov}(\phi_+, \phi_-)$ denotes the covariance of ϕ_+ and ϕ_- . Moreover, we can apply Proposition 3.1 to conclude that

$$\begin{aligned} \text{Cov}(\phi_+, \phi_-) &= \frac{1}{\sigma(K, N)} \sum_{|\underline{l}|=K} \left[\int \phi_+ d\rho_{DS}(\underline{l}) - \int \phi_+ d\mu_N \right] \left[\int \phi_- d\rho_{DS}(\underline{l}) - \int \phi_- d\mu_N \right] \\ &\leq \frac{2\|\phi\|_0}{\sigma(K, N)} \left| \sum_{|\underline{l}|=K} \left[\int \phi_+ d\rho_{DS}(\underline{l}) - \int \phi_+ d\mu_N \right] \right| \\ &\leq \frac{4\|\phi\|_0 V(\phi)}{N} \\ &= \mathcal{O} \left(\frac{1}{N} \right). \end{aligned} \quad (21)$$

The rest of the proof is an immediate consequence of Eqs. (20) and (21). \square

6. CALCULATING THE COEFFICIENTS

Recall that a configuration \underline{l} has $m_j(\underline{l})$ zeroes in (α_{j-1}, α_j) , $1 \leq j \leq N$. Let $0 < p < N$ and let $\{i_1, i_2, \dots, i_p\}$ be a sequence of distinct indices, $1 \leq i_j \leq N$. Finally, let $a_1, a_2, \dots, a_p \in \mathbb{N}$. We want to calculate

$$G(i_1, \dots, i_p; a_1, \dots, a_p) := \sum_{|\underline{l}|=K} \prod_{j=1}^p (m_{i_j}(\underline{l}))^{a_j}. \quad (22)$$

We shall give an answer in terms of the coefficients of the polynomial defined as follows: let $a_1 + a_2 + \dots + a_p = A$. Define a polynomial $P(x) := b_0 + b_1x + b_2x^2 + \dots$ by

$$\prod_{j=1}^p \left[\left(x \frac{d}{dx} \right)^{a_j} \left(\frac{1}{1-x} \right) \right] = \frac{P(x)}{(1-x)^{p+A}}. \quad (23)$$

The polynomial $P(x)$ in (23) is a product of ‘‘elementary’’ polynomials $P_q(x)$ defined by the formula

$$\left(x \frac{d}{dx} \right)^q \left(\frac{1}{1-x} \right) := \frac{P_q(x)}{(1-x)^{q+1}}.$$

We let $P_q(x) = \sum_{m=1}^q b_{q,m} x^m$. In proposition 6.1, we compute the numbers $b_{q,m}$:

Proposition 6.1. *Let $q \geq 1$, and let $b_{q,m}$ be as above. Then*

$$b_{q,m} = \sum_{k=1}^m k^q (-1)^{m-k} \binom{q+1}{m-k}.$$

Proof of Proposition 6.1: By definition,

$$P_q(x) = (1-x)^{q+1} \cdot \left(x \frac{d}{dx}\right)^q \left(\frac{1}{1-x}\right) = (1-x)^{q+1} \left(\sum_{k=1}^{\infty} k^q x^k\right).$$

The proposition now follows by expanding $(1-x)^{q+1}$ and multiplying the two series. \square

We are now in a position to compute the expression $G(i_1, \dots, i_p; a_1, \dots, a_p)$ defined in (22).

Theorem 6.2. *Let b_j -s be the coefficients of a polynomial P defined by (23), and let K be the total number of zeroes in the configuration. Then*

$$G(i_1, \dots, i_p; a_1, \dots, a_p) = \sum_{j=0}^K b_j \sigma(N+A, K-j).$$

Proof of Theorem 6.2: Define a bijection ρ between the set $\{\underline{l} : |\underline{l}| = K\}$ and monomials of degree K in N variables by

$$\rho(\underline{l}) := x_1^{m_1} x_2^{m_2} \dots x_N^{m_N}.$$

Here $K = m_1 + m_2 + \dots + m_N$. We need the following lemma.

Lemma 6.3. *The sum*

$$\sum_{|\underline{l}|=K} \rho(\underline{l})$$

consists of all monomials of total degree K in the infinite series

$$F(x_1, \dots, x_N) := \prod_{i=1}^N \frac{1}{(1-x_i)} = \prod_{i=1}^N (1 + x_i + x_i^2 + \dots).$$

We remark that $F(x_1, \dots, x_N)$ thus defines the generating series for the set of all configurations with N intervals.

Proof of Lemma 6.3: Choose $x_1^{m_1}$ from the first bracket $(1 + x_1 + x_1^2 + \dots)$, $x_2^{m_2}$ from the second bracket etc. \square

If we want to compute the *number* of such monomials, we just let $x_j = x$ and compute the coefficient of x^K in the series expansion of $1/(1-x)^N$. The answer is $\sigma(K, N) = \binom{N+K-1}{N-1}$.

To compute $G(i_1, \dots, i_p; a_1, \dots, a_p)$, we remark that

$$\prod_{j=1}^p \left(x_j \frac{\partial}{\partial x_j}\right)^{a_j} (\rho(\underline{l})) = \prod_{j=1}^p (m_j(\underline{l}))^{a_j} \rho(\underline{l}).$$

Summing over all $|\underline{l}| = K$, we see that we need the sum of coefficients of monomials of degree K in the series

$$\prod_{j=1}^p \left(x_j \frac{\partial}{\partial x_j}\right)^{a_j} F(x_1, \dots, x_N).$$

To compute the sum, we let $x_j = x$; the generating series then becomes equal to

$$\frac{P(x)}{(1-x)^{N+A}},$$

by the definition (23) of $P(x)$. Since the degree of $(1-x)$ in the denominator is equal to $N+A$, we see that the answer will correspond to configurations with $N+A$ intervals. The answer then follows from the multiplication of $P(x)$ by the series $1/(1-x)^{N+A}$. \square

The following computations are used in Section 5.

Corollary 6.4. *For any $1 \leq i \leq N$, $\sum_{|\underline{l}|=K} m_i(\underline{l}) = \sigma(N+1, K-1)$ and $\sum_{|\underline{l}|=K} (m_i(\underline{l}))^2 = \sigma(N+2, K-1) + \sigma(N+2, K-2)$. For any $1 \leq i < j \leq N$, $\sum_{|\underline{l}|=K} m_i(\underline{l})m_j(\underline{l}) = \sigma(N+2, K-2)$.*

Proof of Corollary 6.4: The corollary follows from the Theorem 6.2 and the following two identities:

$$\left(y \frac{\partial}{\partial y}\right) \left(\frac{1}{1-y}\right) = \frac{y}{(1-y)^2} \quad \text{and} \quad \left(y \frac{\partial}{\partial y}\right)^2 \left(\frac{1}{1-y}\right) = \frac{y+y^2}{(1-y)^3}. \quad \square$$

7. ASYMPTOTIC DISTRIBUTION OF ZEROES

There is a rather simple corollary of Theorem 5.1 that we would like to address in this section. Fix a configuration \underline{l} with $|\underline{l}| = K$ and consider the average

$$\bar{\theta}(\alpha; \underline{l}) := \frac{1}{K} \sum_{j=1}^K \theta_j(\alpha; \underline{l}). \quad (24)$$

Recall, a sequence of measurable subsets $U_N; N = 1, 2, \dots$ of a probability space (X, μ) is said to be of full density if $\lim_{N \rightarrow \infty} \mu(U_N) = 1$.

Then, by the Chebyshev inequality it follows from Theorem 5.1 that for any $\alpha \in \Lambda^N$, there exists a full density subset of configurations $D_N(\alpha) \subset \Omega(K, N)$, such that for any $\underline{l} \in D_N(\alpha)$ and $\delta > 0$,

$$\bar{\theta}(\alpha; \underline{l}) = \frac{1}{N+1} \sum_{\nu=0}^N \alpha_\nu + \mathcal{O}_\delta \left(\max\{N^{-1/2+\delta}, K^{-1/2+\delta}\} \right). \quad (25)$$

Let $[0, 1]^\infty$ denote the set of all infinite sequences $\{\alpha_j\}_{j=0}^\infty$ with $\alpha_j \in [0, 1]$. We embed $[0, 1]^{N+1}$ in $[0, 1]^\infty$ via the map $(\alpha_0, \dots, \alpha_N) \mapsto (\alpha_0, \dots, \alpha_N, 0, 0, \dots)$. Similarly, we define $\Lambda^\infty \subset [0, 1]^\infty$ to consist of all strictly increasing sequences with $\alpha_0 < \alpha_1 < \dots$ and $\Lambda^N \subset \Lambda^\infty$ in the same way.

Given any $\sigma \in S_N$, the symmetric group on N -elements, it is clear that the estimate in (25) extends pointwise to $\sigma(\Lambda^N)$ since

$$\frac{1}{N+1} \sum_{\nu=0}^N \alpha_{\sigma(\nu)} = \frac{1}{N+1} \sum_{\nu=0}^N \alpha_\nu.$$

On $\sigma(\Lambda^N)$ the functions $\theta_j(\alpha; \underline{l})$ are defined in the same way as on Λ^N except that $\alpha_{\sigma(j)}$ replaces α_j for $j = 0, \dots, N$. So, the estimate (25) holds Lebesgue a.e. on $[0, 1]^{N+1}$ and consequently, Lebesgue a.e. on $[0, 1]^\infty$ since the $\bar{\theta}(\alpha; \underline{l})$ only depend on the first $N+1$ coordinates $(\alpha_0, \dots, \alpha_N)$. We delete here the set D consisting of points $\alpha = (\alpha_0, \dots, \alpha_N) \in [0, 1]^{N+1}$ where at least two of the α_j 's coincide.

Since the monomials $\alpha_\nu; \nu = 0, \dots, N$ are independent, identically distributed random variables on $[0, 1]^\infty$, by the strong law of large numbers,

$$\frac{1}{N+1} \sum_{\nu=0}^N \alpha_\nu = \frac{1}{2} + \mathcal{O}_\delta(N^{-1/2+\delta}) \quad (26)$$

with probability one on $[0, 1]^\infty$. Combining (25) and (26) gives:

Theorem 7.1. *Let $N, K \rightarrow \infty$. Then, for Lebesgue almost every $\alpha \in [0, 1]^\infty$ there exist a full density of configurations, $D_N(\alpha)$, such that for any $\underline{l} \in D_N(\alpha)$, $|\underline{l}| = K$, and any $\delta > 0$,*

$$\bar{\theta}(\alpha; \underline{l}) = \frac{1}{2} + \mathcal{O}_\delta \left(\max\{N^{-1/2+\delta}, K^{-1/2+\delta}\} \right).$$

Remark: It would be interesting to see whether one can prove a central limit theorem for the $\theta_j(\alpha)$'s. For this, we would need an estimate for $\theta_j(\alpha)$ in terms of the monomials $\alpha_j; j = 0, \dots, N$ that is better than the bound that one gets from the Heine-Stieltjes theorem. We hope to address this in future work.

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