Math 3GP3
Assignment #4

DUE: TUESDAY, NOVEMBER 12TH, 2013

1. The flat Euclidean metric on $\mathbb{C}^2$ can be written as $ds^2 = dz^1\,d\bar{z}^1 + dz^2\,d\bar{z}^2$. Write

$$z^1 = r\cos(\frac{\theta}{2})\exp(\frac{i}{2}(\phi+\psi)) \qquad z^2 = r\sin(\frac{\theta}{2})\exp(\frac{i}{2}(\phi-\psi))$$

and show that the metric on the unit sphere S^3 with $r=1$ is given by $ds^2 = (e^1)^2 + (e^2)^2 + (e^3)^2$, where

$$e^1 + i\,e^2 = \frac{1}{2}e^{i\psi}(d\theta - i\,\sin\theta d\phi) \qquad e^3 = \frac{1}{2}(d\psi + \cos\theta\,d\phi)$$

Calculate the curvature tensor in this basis.

2. Using some computer software, or otherwise, plot graphically a number of solutions $a(t)$ of Friedmann's equations for various values of Λ and κ (all 3 possible signs):

$$\dot{a}^2 = \frac{\Lambda}{3}a^2 - \kappa + \frac{C}{a}$$

where $C = \frac{8\pi}{3}\rho a^3$ is a constant (assuming $p=0$ (dust case)).

3. In a Robertson-Walker model $ds^2 = -dt^2 + a(t)^2 d\sigma_\kappa^2$ of the universe, assume $p=0$ (pure dust with density $\rho > 0$, no pressure) in Friedmann's equations so that ρa^3 is a constant.
Let $H = \frac{\dot{a}}{a}$ (Hubble constant), $q = -\frac{\ddot{a}}{aH^2}$ (deceleration), $J = \frac{1}{aH^3}\frac{d^3a}{dt^3}$ (jerk), $S = \frac{1}{aH^4}\frac{d^4a}{dt^4}$(snap), $\rho_{crit} = \frac{3H^2}{8\pi}$, $\delta = \frac{\rho}{2\rho_{crit}}$ (density parameter), Λ = cosmological constant, κ = spatial curvature.
Establish the following identities:

(i) $\Lambda = 3H^2(\delta - q)$

(ii) $\kappa = a^2H^2(3\delta - q - 1)$

(iii) $J = 3\delta - q$

(iv) $S = -2(\delta + J) - \delta J$