Math 3GP3
Min-Oo
Assignment #2

DUE: TUESDAY, OCTOBER 8TH, 2013
Please hand it to me at the beginning of the lecture period in class

1. Define the operations grad, div and curl in terms of differential forms, exterior differentiation d and the $\star$ operator in $\mathbb{R}^3$. Show that the familiar identities $curl(grad\, f) = 0$, $div(curl\, F) = 0$ follow from $d^2 = 0$.

2. Let $F = (E_x dx + E_y dy + E_z dz) \wedge dt + (B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy)$
and $\star F = dt \wedge (B_x dx + B_y dy + B_z dz) + (E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy)$.

Show that the equations: $dF = 0$ and $d(\star F) = 0$ are equivalent to Maxwell's equations

$$div\vec{B} = 0 \qquad div\vec{E} = 0 \qquad curl\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad curl\vec{B} = \frac{\partial \vec{E}}{\partial t}$$

3. Find an identification of Minkowski space-time with the vector space of 2×2 complex Hermitian matrices $\{X \mid X^* = \bar{X}^t = X\}$ such that negative the determinant of a Hermitian matrix corresponds to the squared length of a vector in space-time.
Show that the action of $g \in SL(2;\mathbb{C})$ = complex 2×2 matrices of determinant 1, on Hermitian matrices given by $X \mapsto g\, X\, g^*$ preserves the Lorentzian metric. Find the Lorentzian transformations corresponding to the following elements of $SL(2;\mathbb{C})$

$$\begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}, \qquad \begin{pmatrix} e^{\frac{\beta}{2}} & 0 \\ 0 & e^{-\frac{\beta}{2}} \end{pmatrix}, \qquad \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

4. Compute the curvature of the 2−dimensional metric, given in polar cordinates by:

$$ds^2 = \frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 \qquad \text{where } \kappa \text{ is a constant.}$$

5. Let $T_{ab} = g^{cd} F_{ac} F_{bd} - \frac{1}{4} F^{cd} F_{cd} g_{ab}$ be the stress-energy tensor in Minkowski space of the electromagnetic field F (satisfying Maxwell's equations)
(i) Show that $\nabla^a T_{ab} \equiv 0$ (*This is a conservation law)*
(ii) Show that $T_{ab} U^a V^b \geq 0$ for any future time-like or null vectors U and V
(*This is known as the dominant energy condition)*
(iii) Show that $T_{ab} \equiv K_a K_b$ for some one form K if and only if $F_{ab} F^{ab} = 0$ and $(\star F)_{ab} F^{ab} = 0$.

6. (*Bonus Question*) Do problem 11.8 on page 91 in the textbook.