Math 3D03
Short solutions to assignment #4

1. *If 5 indistinguishable marbles are placed at random into 5 boxes, what is the probability that* **exactly one** *box is empty?*

There are $\binom{5}{2,1,1,1,0} \times \binom{5}{1,3,1} = \frac{(5!)^2}{2!3!} = 48 \times 5^2$ ways of having exactly one box empty out of all possible 5^5 ways of placing them at random. Therefore the answer is

$$\frac{48}{5^3} = 0.384$$

2. *A point starts at the origin on the real line and takes steps of length δ with probability $p > 0$ to the right and with probability $q = 1 - p$ to the left. Assuming that the steps are independent find the expected value of the* **squared** *distance from the origin after n steps.*

An elegant way to solve this question is to write X as a sum $X = X_1 + \cdots + X_n$, where X_i is the i^{th}-step. Now, $\mathbb{E}[X_i^2] = \delta^2$ for all i and $\mathbb{E}[X_i X_j] = \mathbb{E}[X_i]\mathbb{E}[X_j] = (p-q)^2\delta^2$ for $i \neq j$, and hence $\mathbb{E}[(X_1 + \cdots + X_n)^2] = n\delta^2 + n(n-1)(p-q)^2\delta^2 = n\delta^2\left(4pq + n(p-q)^2\right)$.

A more cumbersome method is to use the binomial distribution:
$\delta^2 \sum_{k=0}^{n}(n-2k)^2\binom{n}{k}p^{n-k}q^k = \delta^2 \sum_{k=0}^{n}(n^2 - 4nk + 4k^2)\binom{n}{k}p^{n-k}q^k$
$= \delta^2\left(n^2 - 4n^2p + 4(npq + n^2p^2)\right) = \left(n^2(p-q)^2 + 4npq\right)\delta^2$
since $\mathbb{E}[X_{bin}] = np$ and $\mathbb{E}[X_{bin}^2] = npq + (np)^2$ for the binomial distribution.

3. *A model for the movement of a stock price supposes that if the present price is S then after one period, it will either go up to uS with probability p or go down to dS with probability $1-p$. Assuming that successive movements are independent, approximate the probability that the stock price will be up by at least 5% after the next 1000 periods for $u = 1.02$, $d = 0.95$ and $p = 0.6$*

The stock price will be up by at least 5% after the next 1000 periods only if there are at least 513 periods where the stock goes up (out of the 1000 periods). This is determined by solving the inequality:

$(1.02)^n(0.95)^{1000-n} \geq 1.05 \iff n \geq \frac{\log(1.05) - 1000\log(0.95)}{\log(1.02) - \log(0.95)} \approx 722.2$

Let X be the number of periods where the stock is going up. Then $X \sim \text{binomial}(1000, 0.6)$.

$P(\text{the stock price will be up by at least } 10\%) = P(X \geq 723)$. By using the normal approximation,

$$P(X \geq 723) \approx P\left(Z \geq \frac{723 - 600}{\sqrt{1000(0.6)(0.4)}}\right) \approx 10^{-15}$$

Fat chance

4. *Do problem 30.18 on page 1215 in the textbook*

The mean is $\frac{a}{2}$ in both cases. The variance is $\frac{a^2}{12}$ in the classical case and is $\frac{a^2}{12} - \frac{a^2}{2\pi^2 n^2}$ in the quantum case. Note that when $n \to \infty$, the quantum variance approaches the classical variance.

5. *Do problem 30.26 on page 1216 in the textbook*

Suppose you have x dollars in your hand. If you play one more time your expected wealth after that is $0.2 \times 0 + 0.5 \times (x+1) + 0.3 \times (x+2) = 1.1 + 0.8\,x$ The optimal strategy therefore is to stop playing when $x \geq 1.1 + 0.8\,x$, i.e. when you have more than 5.5 dollars in your hand and continue otherwise. Let $v(x)$ be the pay-off using this strategy. Then $v(7) = 7$, $v(6) = 6$ and working backwards: $v(5) = 0.5\,v(6) + 0.3\,v(7) = 5.1$, $v(4) = 0.5\,v(5) + 0.3\,v(6) = 4.35$, $v(3) = 0.5\,v(4) + 0.3\,v(5) = 3.705$, $v(2) = 0.5\,v(3) + 0.3\,v(4) = 3.1575$, $v(1) = 0.5\,v(2) + 0.3\,v(3) = 2.68025$, and so finally $v(0) = 0.5\,v(1) + 0.3\,v(2) = 2.587375$ which is the value of the game if you would play the optimal strategy. To fix the number of draws at the beginning is not the optimal strategy, since you would not be using the information that becomes available. However to answer the question in the book, if you draw n times, the probability that you can do all the n draws without getting the blackball is $(0.8)^n$ and hence the expected winnings would be $u(n) = (0.8)^n \times n \times \frac{0.5\times 1 + 0.3\times 2}{0.5+0.3} = 1.1 \times n \times (0.8)^{n-1}$. The maximum value of $u(n)$ is attained at $u(4) = 11 \times 4 \times (0.8)^3 = u(5) = 11 \times 5 \times (0.8)^4 = 2.2528$ which is strictly less than the expected winnings if you are using the optimal strategy, but both values are strictly less than 3, so don't play the game for that fee.

6. *(bonus question)* Two teams A and B play a series of games until one of the teams wins four games (there are no tied games) as in the World Series or the Stanley Cup. Assume that the games are independent and that A wins each game with probability $p > 0$.

(a) Compute the probability that the seventh game is played

(b) What is the expected number of games played?

(c) What is the expected number of games played, given that team A won the series?

(d) Compute the probability that team A won the series, given that the seventh game was played.

(a) The seventh game is played if and only if the series at tied at $3:3$ after the sixth game and the probability of that is $P(7) = \binom{6}{3} p^3 q^3 = 20 p^3 q^3$.

(b) $4 \times (p^4 + q^4) + 5 \times \binom{4}{1}(p^4 q + q^4 p) + 6 \times \binom{5}{2}(p^4 q^2 + q^4 p^2) + 7 \times \binom{6}{3} p^3 q^3$

(c) The probability that A wins the series is $P(A) = p^4 + \binom{4}{1} p^4 q + \binom{5}{2} p^4 q^2 + \binom{6}{3} p^4 q^3$ and the answer is: $\frac{1}{P(A)}\left(4 \times p^4 + 5 \times \binom{4}{1} p^4 q + 6 \times \binom{5}{2} p^4 q^2 + 7 \times \binom{6}{3} p^4 q^3\right)$

(d) The probability that A wins the series at the seventh game is p.