Math 3D03
Short solutions to assignment #3

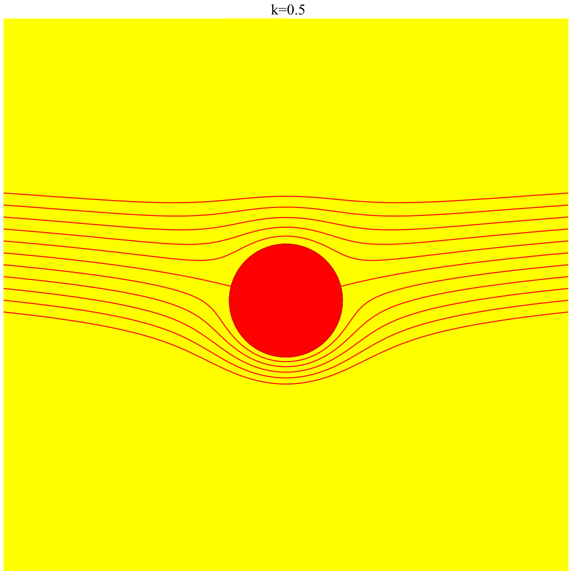
1. Show that
$$w = \tan(z)$$
maps the vertical strip $|x| < \frac{\pi}{4}$ in the z-plane onto the unit disk $|w| < 1$ in the w-plane.

Write $w = \tan(z)$ as a composition of two maps $W = i\,e^{2iz}$ and $w = \frac{W-i}{W+i}$. The first transformation maps the vertical strip $|Re(z)| < \frac{\pi}{4}$ bijectively onto the upper half-plane $Re(W) > 0$ and the second Möbius transformation maps the upper half=plane bijectively onto the unit disk $|w| < 1$.

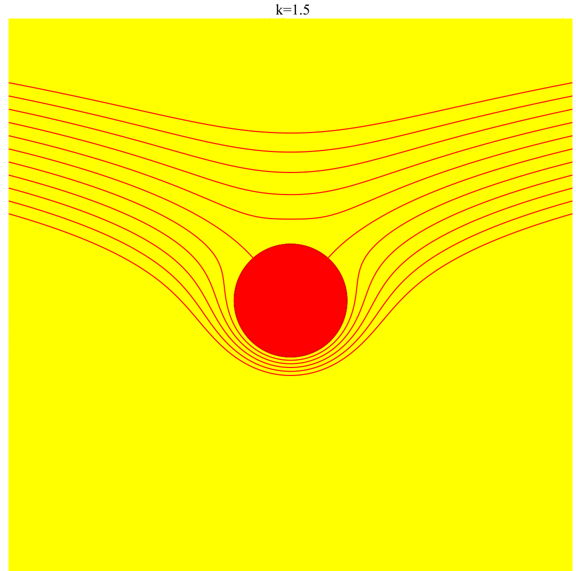
2. The complex potential
$$\Omega(z) = z + \frac{1}{z} - i\,\kappa\log(z)$$
where κ is a positive real number, describes a fluid flow around a cylinder with circulation. Locate the stagnation points (as a function of κ) and sketch the streamlines of the flow, using computer software such as Matlab, for the following κ values: $\kappa = 0.5,\ 1.5,\ 2,\ 3$.

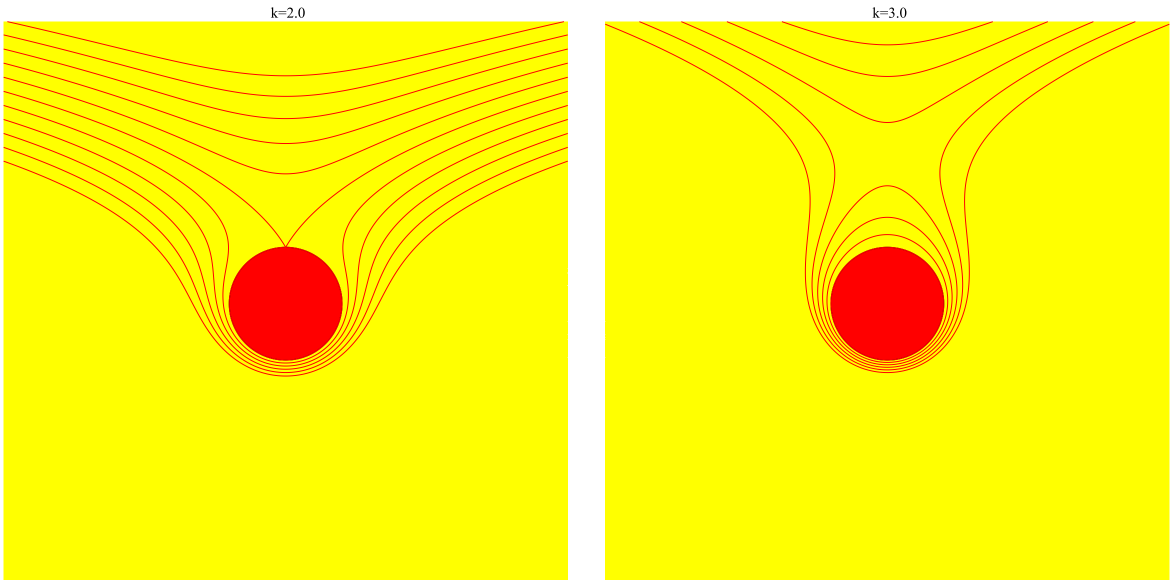
The stagnation points are obtained by solving the quadratic equation $\Omega'(z) = 1 - \frac{1}{z^2} - i\,\kappa\frac{1}{z} = 0$ with roots $\frac{1}{2}\left(i\kappa \pm \sqrt{4-\kappa^2}\right)$. *(I owe the following pictures to L. Ambroszkiewicz)*

k=0.5



k=1.5





3. Find the inverse Laplace transform of

$$\frac{\cosh(x\,s^{\frac{1}{2}})}{s^{\frac{1}{2}}\sinh(a\,s^{\frac{1}{2}})}$$

using a Bromwich contour integral.

We need to integrate $\frac{1}{2\pi i}\oint_C e^{st}\frac{\cosh(x\sqrt{s})}{\sqrt{s}\sinh(a\sqrt{s})}ds$, where C is a large Bromwich semicircle including the negative real axis as your contour. No branch cuts are needed since the function $\frac{\cosh(xz)}{z\ \sinh(az)}$ is an even function of z. The poles are at $z_k = -\frac{\pi^2}{a^2}k^2$ for $k = 0, 1, \ldots$. These are all simple poles with residues given by $\frac{2\ \cosh(x\sqrt{z_k})}{a\cosh(a\sqrt{z_k})}e^{z_k\,t} = (-1)^k\frac{2}{a}\exp(-\frac{\pi^2 k^2}{a^2}\,t)\cos(\frac{\pi x}{a}k)$ for $k \neq 0$ and by $\frac{1}{a}$ for $k = 0$. Therefore the inverse Laplace transform is given by the series:

$$\frac{1}{a} + \frac{2}{a}\sum_{k=1}^{\infty}(-1)^k\exp(-\frac{\pi^2k^2}{a^2}\,t)\cos(\frac{\pi x}{a}k)$$

4. Show that the Airy function:

$$\psi(z) = Ai(z) = \int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}s^3+zs\right)}\,ds$$

satisfies Stokes' equation:

$$\frac{d^2\psi}{dz^2} - z\psi = 0$$

and apply the WKB approximation to obtain the following asymptotic expression for the Airy function as $x \to -\infty$ (x real)

$$Ai(x) \approx \frac{1}{\sqrt{2\pi}}x^{-\frac{1}{4}}\sin\left(\frac{2}{3}|x|^{\frac{3}{2}} + \frac{\pi}{4}\right)$$

Please refer to my notes or the textbook