Math 3D03
Short solutions to assignment #1

1. Compute all values of $i^{(i^i)}$ and $(i^i)^i$

All the values of i^i are given by: $\exp(i\,\log(i)) = \exp(i(i(\frac{\pi}{2}+2k\pi)) = \exp(-\frac{\pi}{2}-2k\pi) \qquad k\in\mathbb{Z}$

so

(i) $i^{(i^i)} = \exp(i^i(\log(i))) = \exp\left(ie^{-\frac{\pi}{2}-2k\pi}(\frac{\pi}{2}+2l\pi)\right) \qquad k,l\in\mathbb{Z}$

(ii) $\left(i^i\right)^i = \exp(i(\log(i^i)) = \exp(-\log(i)) = i^{-1} = -i$

2. Classify all the singular points of the following functions:

$$(a)\;\; f(z) = \frac{\pi\, z}{\sin(\pi z)} \qquad\qquad (b)\;\; f(z) = \frac{z-2}{z^2}\,\sinh\frac{1}{1-z} \qquad\qquad (c)\;\; f(z) = \frac{e^{\frac{1}{z}}}{1-z}$$

(a) The singular points are at $z=\pi k$, for all $k\in\mathbb{Z}$.
$z=0$ is a removable singularity since $\lim_{z\to 0}\frac{\sin\pi z}{\pi z}=1$
All the other singularities $k\pi$ with $k\neq 0$ are simple poles since the numerator is non-zero and the derivative of the denominator is non-zero at all .

(b) $z=0$ is a pole of order 2 and $z=1$ is an essential singularity, since

$\sinh(\frac{1}{1-z}) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(1-z)^{-2k-1}$

(c) $z=1$ is a simple pole and $z=0$ is an essential singularity.

3. Compute the complete Taylor, respectively Laurent series expansion and the region of convergence of the following functions around the point $z=0$:

$$(a)\;\; f(z) = \frac{1}{2i}\,\log\left(\frac{1+iz}{1-iz}\right) \qquad\qquad (b)\;\; f(z) = \frac{e^{\frac{1}{z}}}{1-z}$$

(a) $\frac{1}{2i}\left(\log(1+iz)-\log(1-iz)\right) = \frac{1}{2i}\sum_{k=1}^{\infty}(\frac{(-1)^{k-1}(iz)^k}{k}+\frac{(iz)^k}{k}) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\,z^{2k-1}(=\arctan(z))$.
This Taylor series converges for $|z|<1$.

(b) $\frac{e^{\frac{1}{z}}}{1-z} = \left(\sum_{k=0}^{\infty}\frac{z^{-k}}{k!}\right)\left(\sum_{l=0}^{\infty}z^l\right) = \sum_{n=1}^{\infty}a_{-n}z^{-n} + e\sum_{n=0}^{\infty}z^n$, where $a_{-n}=\sum_{j=n}^{\infty}\frac{1}{j!}$.
This Laurent series converges for $0<|z|<1$.

4. Evaluate the following complex contour integrals:

$$(a)\ \oint_C \frac{dz}{1-z^4} \qquad (b)\ \oint_C \frac{e^{iz}dz}{1+z^2} \qquad (c)\ \oint_C \frac{z^3\,dz}{(z-2)^2(z^2+4)}$$

where C is the ellipse defined by: $3x^2 + 4y^2 = 10^{10}$

(a) The four simple roots of $1 - z^4 = 0$ are given by: $1, i, -1, -i$ on the unit circle all inside the huge ellipse. A partial fraction decomposition gives:

$$\frac{1}{1-z^4} = \frac{1}{4i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) + \frac{1}{4}\left(\frac{1}{z+1} - \frac{1}{z-1}\right)$$

Therefore the required integral vanishes, since the residues cancel in pairs.

(b) The poles are at $\pm i$ (which are inside the huge ellipse) with residues $\frac{e^{-1}}{2i}$ and $\frac{e}{-2i}$ respectively, so the answer is $\pi(e^{-1} - e) = -2\pi\sinh(1)$.

(c) There is a pole of multiplicity 2 at $z = 2$ with residue 1 and simple poles at $z = \pm 2i$ with residues $\frac{-1}{2(i-1)^2}$ and $\frac{-1}{2(i+1)^2}$ respectively, so the answer is $2\pi i$.

Remark: These integrals can also be evaluated by computing the residue at ∞ after making a substitution $w = \frac{1}{z}$ into the "one-form" $f(z)dz$. For example (c) can be evaluated by integrating $\frac{-dw}{w(1-2w)^2(1-4w^2)}$ clockwise around a small circle $|w| = 100^{-100}$.

5. Compute the coefficient of z^3 in the power series expansion (around $z = 0$) of $(T(z))^4$, where

$$T(z) = \frac{z}{1 - e^{-z}}$$

There are many ways to calculate this, but the answer is 1.