MATH 3C03
Short Answers to Test # 1

1. State whether the following statements are TRUE or FALSE *(no explanations needed)*

(i) Every square matrix is diagonalisable. FALSE

(iii) For any matrix A, all the non-zero eigenvalues of AA^T and A^TA are the same. TRUE

(iv) All eigenvalues of a Hermitian matrix are real numbers. TRUE

(v) For any square matrix A, $det(e^A) \neq 0$ TRUE

(vi) $x = \infty$ is a regular singular point of Hermite's eqn. $y''(x) - 2xy'(x) + 2\lambda y(x) = 0$. FALSE

(vii) There is a quadratic polynomial $x(t)$ satisfying $(1-t^2)\,\ddot{x}(t) - 2t\,\dot{x}(t) + 12\,x(t) = 0$ FALSE

2. Find the characteristic frequencies and the normal modes of the coupled harmonic oscillators:

$$\begin{aligned} \ddot{x}_1 &= -x_1 - 4\,(x_1 - x_2) \\ \ddot{x}_2 &= -x_2 - 4\,(x_2 - x_1) \end{aligned}$$

Answer: The slow characteristic frequency is $\omega_1 = \sqrt{1} = 1$ with normal mode (eigenfunction) $(1,1)^T$ and the fast characteristic frequency $\omega_2 = \sqrt{9} = 3$ with normal mode with (eigenfunction) $(+1,-1)^T$

3 (i) Find the Fourier series of the odd rectangular wave function $sgn(x) = \frac{x}{|x|}$, $-\pi \leq x \leq +\pi$, periodically extended with period 2π.
(ii) Use this series to show that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

Answer: The step function $f(x) = sgn(x) = \frac{x}{|x|}$, $-\pi < x \leq \pi$, periodically extended, is odd with Fourier sine series:

$$sign(x) \approx \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{2k-1}$$

Parseval's identity shows that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

4. Find the Fourier transform of the function:

$$f(t) = e^{-|t|} \qquad\qquad -\infty < t < +\infty$$

and verify that Parseval's identity (Plancherel's Theorem) holds in this case.

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-|t|} e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{(1-i\omega)t} dt + \int_{0}^{+\infty} e^{(-1-i\omega)t} dt \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\omega} + \frac{1}{1+i\omega} \right) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}$

$\int_{-\infty}^{+\infty} |f(t)|^2 = 2 \int_{0}^{+\infty} e^{-2t} dt = 1 = \int_{-\infty}^{+\infty} |f(\omega)|^2 d\omega = \frac{4}{\pi} \int_{0}^{+\infty} \frac{d\omega}{(1+\omega^2)^2}$ since putting $\omega = \tan\theta$ gives:

$\int_{0}^{+\infty} \frac{d\omega}{(1+\omega^2)^2} = \int_{0}^{\frac{\pi}{2}} \cos^2\theta d\theta = \frac{\pi}{4}$

5. Compute

$$\int_{-\infty}^{+\infty} \frac{\sin^2(\omega)}{\omega^2} \, d\omega$$

by using Fourier transforms

The Fourier transform of the rectangular wave function $rect(t)$ of height 1 from $-1 \leq t \leq +1$ is $\sqrt{\frac{2}{\pi}}\, sinc(\omega)$, so by Parseval's identity:

$$\int_{-\infty}^{+\infty} \frac{\sin^2(\omega)}{\omega^2} \, d\omega = \pi$$

6 (*bonus*). Find a polynomial solution of the differential equation:

$$(1 - t^2)\, \ddot{x}(t) - 2t\, \dot{x}(t) + 12\, x(t) = 0$$

Answer: $P_3(t) = \frac{1}{2}(5t^3 - 3t)$