Math 3C03
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Short Answers to Assignment #5

1. Solve the heat equation

$$\frac{\partial}{\partial t}u(x,t) = \frac{1}{\kappa^2}\frac{\partial^2}{\partial x^2}u(x,t)$$

on the real line $\mathbb{R}$ with initial condition:

$$u(x,0) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The solution is given by the convolution with the Heat Kernel as I explained in class.

$$\begin{aligned} u(x,t) &= \frac{\kappa}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} u(\xi,0)\exp\left(-\frac{\kappa^2(x-\xi)^2}{4t}\right)d\xi \\ &= \frac{\kappa}{\sqrt{4\pi t}}\int_{-1}^{+1} \exp\left(-\frac{\kappa^2(x-\xi)^2}{4t}\right)d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{z_-}^{z_+} e^{-\frac{z^2}{2}}\,dz = \Phi(z_+) - \Phi(z_-) \end{aligned}$$

where $z_\pm = \frac{\kappa}{\sqrt{2t}}(\pm 1 - x)$.

2. (i) Find the (Dirichlet) Green's function for the quadrant $Q = \{(x_1,x_2)|\, x_1 \ge 0, x_2 \ge 0\}$ in $\mathbb{R}^2$

(ii) Solve the Dirichlet problem:

$$\Delta u = 0 \text{ in } Q \;\text{ with } u(x,0) = f(x)\,,\; u(0,y) = g(y) \text{ on } \partial Q$$

(i) Using the method of images, we need three points reflected across the boundary outside of Q:

$$q = (x_1,x_2) \in Q,\; q^{\pm} = (x_1,-x_2),\; q^{\mp} = (-x_1,x_2),\; -q = (-x_1,-x_2)$$

The Green's function for Q is then simply a sum of four terms:

$$\bar{G}(p,q) - \bar{G}(p,q^{\pm}) - \bar{G}(p,q^{\mp}) + \bar{G}(p,-q)$$

where $\bar{G}(p,q) = \frac{1}{2\pi}\log(|p-q|) = \frac{1}{\pi}\log(|p-q|^2)$.

(ii) To solve the Dirichlet problem, we need to find the normal derivatives at the boundaries, which are simply $\frac{\partial}{\partial y}$ on the $x-$axis and $\frac{\partial}{\partial x}$ on the $y-$axis respectively. We "plug that in" into Green's formula to get:

$$\begin{aligned} u(x_1,x_2) = \;& \frac{1}{\pi}\int_0^{\infty} f(\xi)\left(\frac{x_2}{(\xi-x_1)^2+x_2^2} - \frac{x_2}{(\xi+x_1)^2+x_2^2}\right)d\xi \\ +\;& \frac{1}{\pi}\int_0^{\infty} g(\eta)\left(\frac{x_1}{x_1^2+(\eta-x_2)^2} - \frac{x_1}{x_1^2+(\eta+x_2)^2}\right)d\eta \end{aligned}$$

3. Do problem 21.28 on page 773 in the textbook.

By the product rule:

$$\begin{aligned}\nabla \cdot (p\phi\nabla\psi - p\psi\nabla\phi) &= p\phi\nabla^2\psi + \phi\nabla p \cdot \nabla\psi + p\nabla\phi \cdot \nabla\psi - p\psi\nabla^2\phi - \psi\nabla p \cdot \nabla\phi - p\nabla\psi \cdot \nabla\phi \\ &= \phi p\nabla^2\psi + \phi\nabla p\nabla\psi + \phi q\psi - \psi p\nabla^2\phi - \psi\nabla p\nabla\phi - \psi q\phi \\ &= \phi\mathcal{L}\psi - \psi\mathcal{L}\phi\end{aligned}$$

Now apply Green's (or divergence) theorem.

4. Do problem 19.8 on page 672-673 in the textbook.

By the product rule for commutators:

$$[x_n, p_{x_n}^2] = [x_n, p_{x_n}]p_{x_n} + p_{x_n}[x_n, p_{x_n}] = 2i\hbar p_{x_n}$$

Now since each x_n commutes with everything in sight **except** with p_{x_n}, we get:

$$[x, H] = \frac{1}{2m}\sum_{n=1}^{N}[x_n, p_{x_n}^2] = \frac{i\hbar}{m}\sum_{n=1}^{N} p_{x_n}$$

and hence:

$$L = [[x, H], x] = \frac{i\hbar}{m}\sum_{n=1}^{N}[p_{x_n}, x_n] = N\frac{\hbar^2}{m}$$

Expressing in terms of a **complete** basis of eigenstates $|r>$ of H with eigenvalues E_r:

$$< r_1|[x, H]|r_2 >=< r_1|xE_{r_2} - E_{r_1}x|r_2 >$$

and so

$$\begin{aligned}N\frac{\hbar^2}{2m} &= \frac{1}{2} < 0|[[x, H]x]|0 > \\ &= \frac{1}{2}\sum_{k=0}^{\infty}(< 0|(xE_k - E_0x)|k >< k|x|0 > - < 0|x|k >< k|(xE_0 - E_kx)|0 >) \\ &= \sum_{k=0}^{\infty}(E_k - E_0) < k|x|0 >^2\end{aligned}$$

5. Do problem 22.26 on page 800 in the textbook.

First of all, $J_n = \int_0^\infty r^n \exp(-2\beta r)dr = \Gamma(n+1)(2\beta)^{-(n+1)}$ and so for $\psi = \exp(-\beta r)$, we have

$$|\psi|^2 = vol(S^2)\int_0^\infty r^2 \exp(-2\beta r)dr = \pi\beta^{-3}$$

and $<\psi\,|H|\,\psi> = \frac{\hbar^2}{2m}\int|\nabla\psi|^2 - \frac{q^2}{4\pi\varepsilon_0}\int\frac{1}{r}|\psi|^2$, where we integrate on all of $\mathbb{R}^3$.

$$\int|\nabla\psi|^2 = 4\pi\int_0^\infty r^2(-\beta\exp(-\beta r))^2dr = \pi\beta^{-1} \quad\text{and}\quad \int\frac{1}{r}|\psi|^2 = 4\pi\int_0^\infty r\exp(-2\beta r)dr = \pi\beta^{-2}$$

and hence

$$\frac{<\psi\,|H|\,\psi>}{|\psi|^2} = \frac{\hbar^2}{2m}\beta^2 - \frac{q^2}{4\pi\varepsilon_0}\beta$$

which is quadratic in β and has a minimum value of

$$-\frac{mq^4}{2(4\pi\varepsilon_0\hbar)^2}$$

at $\beta = \frac{mq^2}{4\pi\varepsilon_0\hbar^2}$

As we know from the lectures, this is in fact the exact value of the lowest energy for the hydrogen atom (Bohr model).

6. (*bonus question*) Consider two independent quantum harmonic oscillators with annihilation/creation operators $A_1, A_2, A_1^\dagger, A_2^\dagger$, satisfying the commutation relations:

$$[A_i, A_j] = [A_i^\dagger, A_j^\dagger] = 0\,,\ [A_i, A_j^\dagger] = \hbar\,\delta_{ij} \qquad\qquad i,j = 1,2$$

with vacuum state $|0\rangle$ satisfying $A_1|0\rangle = A_2|0\rangle = 0$ and with normalized eigenstates:

$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1!\,n_2!}}(A_1^\dagger)^{n_1}(A_2^\dagger)^{n_2}\,|0\rangle$$

containing n_1 excitations of the first harmonic oscillator and n_2 of the second.
Define the operators:

$$J_+ = A_1^\dagger\,A_2\,,\ J_- = A_2^\dagger\,A_1\,,\ J_0 = \frac{1}{2}(A_1^\dagger\,A_1 - A_2^\dagger\,A_2)\,,\ N = A_1^\dagger\,A_1 + A_2^\dagger\,A_2$$

(i) Compute the commutation relations between the operators: $J_\pm, J_0, N$

(ii) Compute $J_\pm|n_1, n_2\rangle$, $J_0|n_1, n_2\rangle$ and express the result in terms of the half integral quantum numbers $j = \frac{1}{2}(n_1 + n_2)$, $m = \frac{1}{2}(n_1 - n_2)$

“JUST DO IT” *NIKE*

Putting $\hbar = 1$ for simplicity, you will find that $J_\pm, J_0$ satisfy the Lie algebra of $su(2)$:

$$[J_0, J_\pm] = \pm J_\pm\ ,\ [J_+, J_-] = 2J_0$$

and N commutes with everything. This shows that the rotation algebra can be thought of as two independent harmonic oscillators.