Math 3C03

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Assignment #5

Due: Thursday, November 21st, 2013 in class at the beginning of the lecture

1. Solve the heat equation

$$\frac{\partial}{\partial t}u(x,t) = \frac{1}{\kappa^2}\frac{\partial^2}{\partial x^2}u(x,t)$$

on the real line $\mathbb{R}$ with initial condition:

$$u(x,0) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

2. (i) Find the (Dirichlet) Green's function for the quadrant $Q = \{(x_1, x_2)|\, x_1 \geq 0, x_2 \geq 0\}$ in $\mathbb{R}^2$

(ii) Solve the Dirichlet problem:

$$\Delta u = 0 \text{ in } Q \;\text{ with } u(x,0) = f(x)\,,\; u(0,y) = g(y) \text{ on } \partial Q$$

3. Do problem 21.28 on page 773 in the textbook.

4. Do problem 19.8 on page 672-673 in the textbook.

5. Do problem 22.26 on page 800 in the textbook.

6. (*bonus question*) Consider two independent quantum harmonic oscillators with annihilation/creation operators $A_1, A_2, A_1^\dagger, A_2^\dagger$, satisfying the commutation relations:

$$[A_i, A_j] = [A_i^\dagger, A_j^\dagger] = 0\,,\; [A_i, A_j^\dagger] = \hbar\,\delta_{ij} \qquad\qquad i,j = 1,2$$

with vacuum state $|0\rangle$ satisfying $A_1|0\rangle = A_2|0\rangle = 0$ and with normalized eigenstates:

$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1!\, n_2!}}(A_1^\dagger)^{n_1}(A_2^\dagger)^{n_2}\,|0\rangle$$

containing n_1 excitations of the first harmonic oscillator and n_2 of the second.
Define the operators:

$$J_+ = A_1^\dagger\, A_2\,,\; J_- = A_2^\dagger\, A_1\,,\; J_0 = \frac{1}{2}(A_1^\dagger\, A_1 - A_2^\dagger\, A_2)\,,\; N = A_1^\dagger\, A_1 + A_2^\dagger\, A_2$$

(i) Compute the commutation relations between the operators: $J_\pm, J_0, N$

(ii) Compute $J_\pm|n_1, n_2\rangle\,,\;\; J_0|n_1, n_2\rangle$ and express the result in terms of the half integral quantum numbers $j = \frac{1}{2}(n_1 + n_2)\,,\, m = \frac{1}{2}(n_1 - n_2)$