Math 3C03
Short Answers to Assignment #3

#1. Laguerre's differential equation is given by:

$$z\frac{d^2y}{dx^2} + (1-z)\frac{dy}{dz} + \lambda y = 0.$$

Using $p(z) = (1-z)/z$ and $q(z) = \lambda/z$, we obtain:

$$\lim_{z\to 0} zp(z) = 1 \quad \text{and} \quad \lim_{z\to 0} z^2 q(z) = 0.$$

Hence, $z = 0$ is a regular singular point, and we can use Frobenius method to find a power series solution of the form

$$y(z) = \sum_{n=0}^{\infty} a_n z^{n+\sigma}.$$

Substituting in the ODE and rearranging terms, we get:

$$z^\sigma \left[\sum_{k=-1}^{\infty} (k+1+\sigma)^2 a_{k+1} z^k - \sum_{k=0}^{\infty} (k+\sigma-\lambda) a_k z^k \right] = 0,$$

$$z^\sigma \left[[\sigma(\sigma-1)+\sigma] a_0 z^{-1} + \sum_{k=0}^{\infty} \left[(k+1+\sigma)^2 a_{k+1} - (k+\sigma-\lambda) a_k \right] z^k \right] = 0.$$

The indicial equation is:

$$\sigma(\sigma-1) + \sigma = \sigma^2 = 0 \quad \Rightarrow \quad \sigma_{1,2} = 0.$$

The recurrence relation is:

$$a_{k+1} = \frac{k+\sigma-\lambda}{(k+\sigma+1)^2} a_k = \frac{k-\lambda}{(k+1)^2} a_k$$

for $\sigma = 0$. For $\lambda = N$, N non-negative integer, $a_j \equiv 0$, for $j = N+1, N+2, \ldots$ Therefore, the solutions are polynomials of order N given by:

$$L_N(z) = \sum_{n=0}^{N} a_n z^n.$$

Using the recurrence relation, the n-th coefficient is given by:

$$a_n = a_0 \prod_{k=1}^{n} \frac{-(N-(k-1))}{k^2} = a_0 \frac{(-1)^n N!}{(N-n)!(n!)^2}, \quad \text{therefore}$$

$$L_N(z) = \sum_{n=0}^{N} \frac{(-1)^n (N!)^2}{(N-n)!(n!)^2} z^n,$$

since $L_N(0) = a_0 = N!$.

#2. We want to find constants c_n, $n = 0, 1, \dots, 4$ such that

$$f(x) = x^2(1-x)^2 = x^4 - 2x^3 + x^2 \sum_{n=0}^{4} c_n P_n(x)$$

where P_n is the Legendre polynomial of order n. The coefficients c_n are given by:

$$c_n = \frac{\int_{-1}^{1} f(x)P_n(x)dx}{\int_{-1}^{1} P_n(x)P_n(x)dx} = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$$

We find that:

$$c_0 = \frac{8}{15}, \quad c_1 = -\frac{6}{5}, \quad c_2 = \frac{26}{21}, \quad c_3 = -\frac{4}{5}, \quad c_4 = \frac{8}{35}.$$

This can also be obtained by equating coefficients of the two quartic polynomials. The equivalent of Parseval's Theorem can be stated as:

$$\begin{aligned} \int_{-1}^{1} |f(x)|^2 dx &= \int_{-1}^{1} \left(\sum_{n=0}^{4} c_n P_n(x)\right)\left(\sum_{m=0}^{4} c_m P_m(x)\right) dx \\ &= \sum_{n=0}^{4} \frac{2}{2n+1} c_n^2. \end{aligned}$$

Evaluating the integral and the finite sum we obtain 736/315 for both of them, hence verifying Parseval's identity.

#3. The potential from the"north pole" is given by:

$$\Phi^+ = \frac{-q}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0}^{\infty} P_n(\cos\theta)(\frac{a}{r})^n$$

from the"south pole" by:

$$\Phi_- = \frac{-q}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0}^{\infty} P_n(-\cos\theta)(\frac{a}{r})^n = \frac{-q}{4\pi\epsilon_0}\frac{1}{r}\sum_{n=0}^{\infty} (-1)^n P_n(\cos\theta)(\frac{a}{r})^n$$

and from the origin by $\Phi_0 = \frac{2q}{4\pi\epsilon_0}\frac{1}{r}$. Adding up the three contributions, we get:

$$\Phi = \frac{-q}{4\pi\epsilon_0}\frac{1}{r}\sum_{k=1}^{\infty} P_{2k}(\cos\theta)(\frac{a}{r})^2 k \qquad \text{(only positive even powers)}$$

#4. Put $h = i\, e^{i\theta}$ in the given generating function. We get:

$$\exp(iz\cos\theta) = \exp\left(\frac{z}{2} i(e^{i\theta} + e^{-i\theta})\right) = \sum_{k=-\infty}^{\infty} J_k(z) i^k e^{ik\theta}$$

which is a Fourier series. Now use $J_n(z) = (-1)^n J_n(z)$; equate real and imaginary parts, and use the formula for Fourier coefficients to get:

$$(-1)^m J_{2m}(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos(z\cos\theta)\cos(2m\theta) \qquad (-1)^m J_{2m+1}(z) = \frac{1}{2\pi}\int_0^{2\pi} \sin(z\cos\theta)\cos((2m+1)\theta)$$

#5. The integral is equal to $\|xH_n(x)\|^2$ with respect to the inner product $< f|g >= \int_{-\infty}^{+\infty} f(x)g(x)e^{-x^2}dx$. We use the following facts:

(i) Hermite polynomials are orthogonal with respect to this inner product

(ii) $\|H_k(x)\|^2 = 2^k k!\sqrt{\pi}$ (normalisation)

(iii) the recursion formula $xH_n(x) = \frac{1}{2}H_{n+1}(x) + nH_{n-1}(x)$.

$$\begin{aligned}
\|xH_n(x)\|^2 &= \|\frac{1}{2}H_{n+1}(x) + nH_{n-1}(x)\|^2 \\
&= \frac{1}{4}\|H_{n+1}(x)\|^2 + n^2\|H_{n-1}(x)\|^2 \\
&= \sqrt{\pi}\left(2^{n-1}(n+1)! + n\,2^{n-1}n!\right) \\
&= (n+\frac{1}{2})\sqrt{\pi}\,2^n\,n!
\end{aligned}$$

#6 (*bonus question*) Proof by contradiction:
Let $\alpha < \beta$ be two consecutive zeros of y_1 and let's assume that $y_2 > 0$ in the interval $[\alpha, \beta]$. We can also assume without loss of generality that $y_1 > 0$ in the open interval (α, β) and so $y_1'(\alpha) > 0$, $y_1'(\beta) < 0$. Now

$$\begin{aligned}
0 &= \int_\alpha^\beta (y_1'' + q_1 y_1)y_2 - (y_2'' + q_2 y_2)y_1 \\
&= \int_\alpha^\beta (q_1 - q_2)y_1 y_2 + \left[(y_1')y_2 - y_2' y_1\right]_{x=\alpha}^{x=\beta}
\end{aligned}$$

which is a contradiction since $\int_\alpha^\beta (q_1 - q_2)y_1 y_2 < 0$
and also the boundary term $[(y_1')y_2 - y_2'y_1]_{x=\alpha}^{x=\beta} = y_1'(\beta)y_2(\beta) - y_1'(\alpha)y_2(\alpha) < 0$