**Math 3C03**
**Short Answers to Assignment #2**

#1.

$$|x| \;\approx\; \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos((2k-1)x)}{(2k-1)^2}$$

This can be computed directly or by integrating the step function $sgn(x) = \frac{x}{|x|}$, $-\pi < x \leq \pi$,

$$sgn(x) \;\approx\; \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{2k-1}$$

as was done in my notes. Integrating once more we get:

$$\frac{1}{2} sgn(x) x^2 \;\approx\; \frac{\pi}{2}x - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{(2k-1)^3}$$

and therefore

$$\frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{(2k-1)^3} = \frac{\pi}{2}x - \frac{1}{2} sgn(x) x^2 = g(x)$$

and setting $x = \frac{\pi}{2}$ now gives

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)^3} = \frac{\pi^3}{32}$$

#2.

$$\hat{\psi}(0,0,k) = C\int_0^{2\pi}\int_{-\pi}^{+\pi}\int_0^{\infty} \exp(-ikr\cos\theta) r\cos\theta \exp(-\frac{r}{2a_0}) r^2 \sin\theta dr d\theta d\phi$$
$$= 2\pi C\int_0^{\infty} \exp(-\frac{r}{2a_0}) r^3 \int_{-1}^{+1} u \exp(-ikru) du\, dr$$

where we used the substitution $u = \cos\theta$ and $2\pi C = \frac{1}{4} a_0^{-\frac{5}{2}}$ Now since

$$\int_{-1}^{+1} u \exp(-ikru) du = (e^{+ikr} - e^{-ikr})(\frac{1}{ikr} - \frac{1}{k^2 r^2})$$

$$\hat{\psi}(0,0,k) = \frac{1}{4} a_0^{-\frac{5}{2}} \int_0^{\infty} (\exp(\alpha_k r) - \exp(\bar{\alpha}_k r))\,(\frac{r^2}{ik} - \frac{r}{k^2}) dr$$

where $\alpha_k = \frac{-1}{2a_0} + ik \qquad \bar{\alpha}_k = \frac{-1}{2a_0} - ik$. After a few manipulations (without using Wolfram alpha!), I find:

$$\hat{\psi}(0,0,k) = 8a_0^{\frac{3}{2}} \left[ \frac{12 - 8a_0^2 k^2}{(1 + 4a_0^2 k^2)^3} - \frac{i}{a_0 k (1 + 4a_0^2 k^2)^2} \right]$$

#3. The Fourier transform of the basic square wave is the *sinc* function and of the tent (or triangular) wave is $sinc^2$ function (as was done in class). The questions here can be solved by using the properties of Fourier transforms (superposition, dilation, shifts, scaling, convolution, etc.). The answers are as follows:

(a)

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{2}e^{-\frac{1}{2}i\omega}sinc^2(\frac{1}{4}\omega)+2e^{-2i\omega}sinc^2(\frac{1}{2}\omega)+\frac{9}{2}e^{-\frac{9}{2}i\omega}sinc^2(\frac{3}{4}\omega)\right)$$

(b)

$$\frac{1}{\sqrt{2\pi}}\left(8sinc^2(\omega)-2sinc^2(\frac{1}{2}\omega)\right)$$

(c)

$$\frac{1}{\sqrt{2\pi}}\left((1+i\omega)e^{-i\omega}sinc^2(\frac{1}{2}\omega)\right)$$

#4.

$$T_e=\frac{1}{x(0)}\int_{-\infty}^{+\infty}x(t)e^{-i0t}dt=\frac{\sqrt{2\pi}\,\hat{x}(0)}{x(0)}$$

and

$$B_e=\frac{1}{\hat{x}(0)}\int_{-\infty}^{+\infty}\hat{x}(t)e^{+i0t}dt=\frac{\sqrt{2\pi}\,x(0)}{\hat{x}(0)}$$

Therefore $T_e\,B_e=2\pi$.

For the function $x(t)=exp(-\frac{|t|}{T}), T_e=\int_{-\infty}^{+\infty}exp(-\frac{|t|}{T})dt=2T$ and hence $B_e=\frac{\pi}{T}$ and the Fourier transform is:

$$\hat{x}(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-\frac{t}{T}}\cos(\omega t)dt=\sqrt{\frac{2}{\pi}}\frac{T}{1+\omega^2T^2}$$

Note that this is in fact a Laplace transform! The total energy is

$$\int_{-\infty}^{+\infty}|x(t)|^2dt=\int_{-\infty}^{+\infty}|x(\omega)|^2d\omega=T$$

and so

$$\frac{1}{T}\frac{2T^2}{\pi}\int_{-\frac{\pi}{4T}}^{+\frac{\pi}{4T}}\frac{1}{(1+(\omega T)^2)^2}d\omega=\frac{8}{16+\pi^2}+\frac{2}{\pi}arctan(\frac{\pi}{4})\approx 0.73$$

#5. Taking the Laplace transform of the differential equation

$$\begin{aligned}\ddot{x}+2x+y &= \cos(t)\\ \ddot{y}+2x+3y &= 2\cos(t)\end{aligned}$$

with initial values $x(0) = \dot{x}(0) = y(0) = \dot{y}(0)$, we obtain:

$$\begin{aligned} s^2F(s) + 2F(s) + G(s) &= \frac{s}{s^2+1} \\ G(s) + 2F(s) + 3G(s) &= \frac{2s}{s^2+1} \end{aligned}$$

where $F$ and $G$ are the Laplace transforms of $x$ and $y$ respectively.
Solving for $F$ and $G$, we get:

$$G(s) = 2F(s) = \frac{2s}{(s^2+4)(s^2+1)} = \frac{2}{3}\left(\frac{s}{s^2+1} - \frac{s}{s^2+4}\right)$$

and hence

$$y(t) = 2x(t) = \frac{2}{3}\left(\cos(t) - \cos(2t)\right)$$

The system therefore moves on the line $y = 2x$.

The matrix $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ has eigenvalues $\omega^2 = 1, 2$ with corresponding eigenvectors $(1, -1)^T$ and $(1, 2)^T$. The frequency of the driving force is 1, whereas the system is moving in the eigen-direction (normal mode) of the second eigenvalue 2, perpendicular to the normal mode of the first eigen-frequency and hence there is no resonance.

#6 (*bonus question*)
The Fourier transform of a single strip from $a$ to $2a$ is $\hat{f}_1(q) = \frac{A}{\sqrt{2\pi}}\left(\frac{e^{-iaq} - e^{-2iaq}}{iq}\right)$, where $q = k\sin\theta$. A shift of $2an$ units (to the right) changes the Fourier transform by a factor of $e^{-2ianq}$. Therefore by linearity, the Fourier transform of the grating is:

$$\hat{f}_1(q)\sum_{n=-N}^{N-1} e^{-2ianq} = \hat{f}_1(q)\frac{e^{2iaNq} - e^{-2iaNq}}{1 - e^{-2iaq}} = \frac{A}{\sqrt{2\pi}}e^{-\frac{1}{2}iaq}\,\frac{\sin(2aNq)}{q\cos(\frac{1}{2}aq)}$$

The intensity $I(\theta)$ is proportional to

$$|\hat{f}(q)|^2 = \frac{A^2}{2\pi}\,\frac{\sin^2(2aNq)}{q^2\cos^2(\frac{1}{2}aq)}$$

For large $N$, $\sin^2(2aNq)$ oscillates very rapidly and uniformly between 0 and 1.

For small $q$, $\sin^2(2aNq) \approx 4a^2N^2q^2$ and $\cos^2(\frac{1}{2}aq) \approx 1$, so $|\hat{f}(q)|^2 \approx \frac{2}{\pi}A^2a^2N^2$ .

For $\frac{1}{2}aq \approx (2m+1)\frac{\pi}{2}$ , we can use L'Hôpital's rule to get

$$|\hat{f}(q)|^2 \approx \frac{8A^2a^2N^2}{(2m+1)\pi^3}$$

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